# Contents

1 Shells in nature and engineering

2 The characteristics of shells as a structural elements
- Geometrical description
- Load carrying mechanism – membrane forces
- Loss of stability phenomenon

3 Examples of shells used in mechanics and biomechanics – structures, materials, manufacturing

# Definition of shell

## Shell

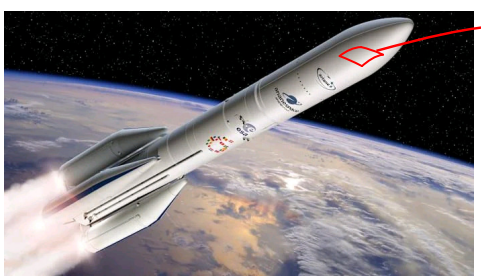
structural element the thickness of which is much smaller than the other two dimensions

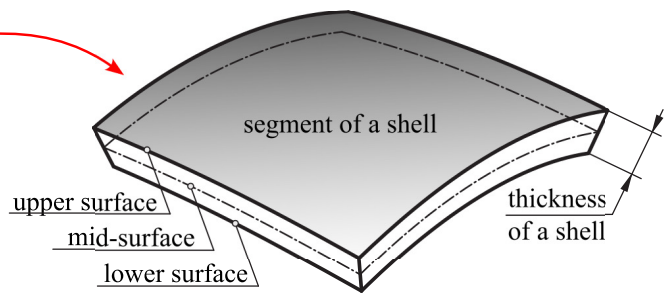
## Middle surface

Locus of points that lie at equal distance from upper and lower surface

## Thickness of a shell

distance length of the segment, perpendicular to the surface





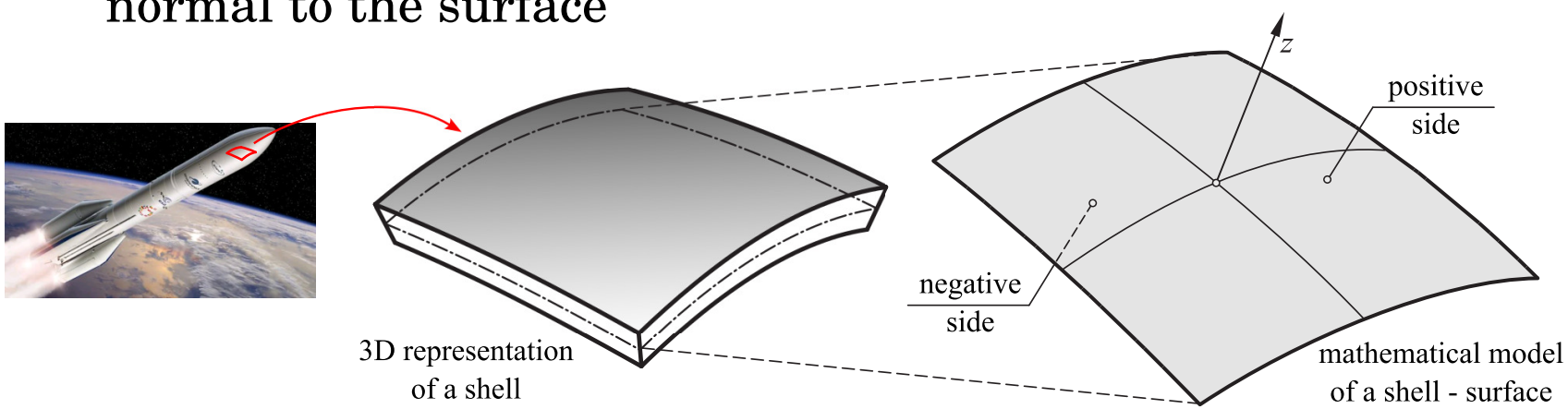
# Definition of shell

## Mathematical model of a shell

### Surface

the boundary between two distinct regions of three-dimensional Euclidean space

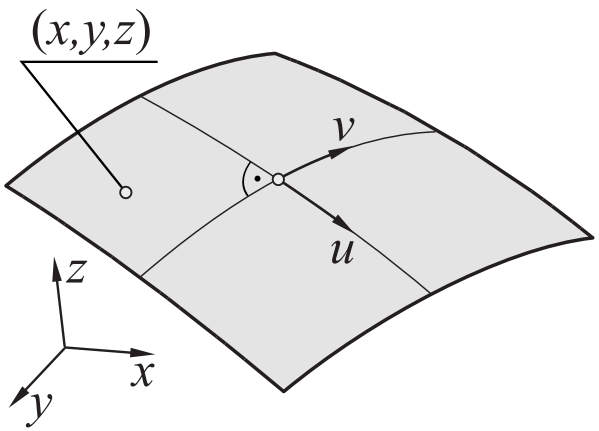
- by definition, a surface has zero thickness
- it separates this what is above from this what is below
- to make some order, we introduce the coordinate $z$, which is normal to the surface



# Definition of shell

## Dimensionality of a surface

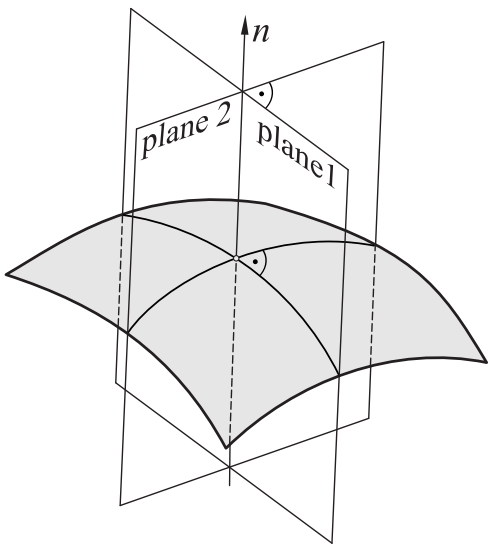
- an interesting aspect is the dimensionality of a surface
- each point on a surface is defined by three Cartesian coordinates $(x, y, z)$
- the position of each point on the surface can be also determined by two coordinates $(u, v)$ defined along two perpendicular directions
- this makes the surface two dimensional object in three dimensional space (similar to the surface of the Earth, described by two directions)

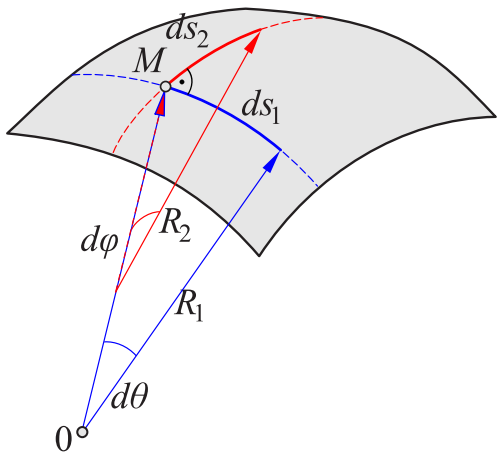


# Definition of shell

## Description of a shell

- we are going to describe geometry of a shell along the curves lying on it
- the curves are made by intersection of the shell with two perpendicular planes passing through the normal to the shell



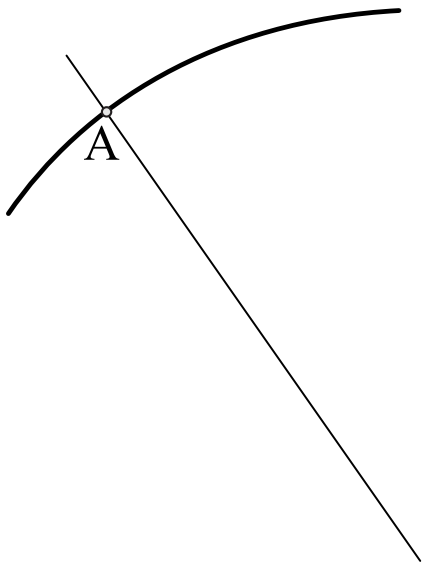




- thus, to describe the shell basic information are required regarding a place curve

# Definition of shell

## Description of a plane curve

- let's draw a plane curve and the normal to that curve at point A

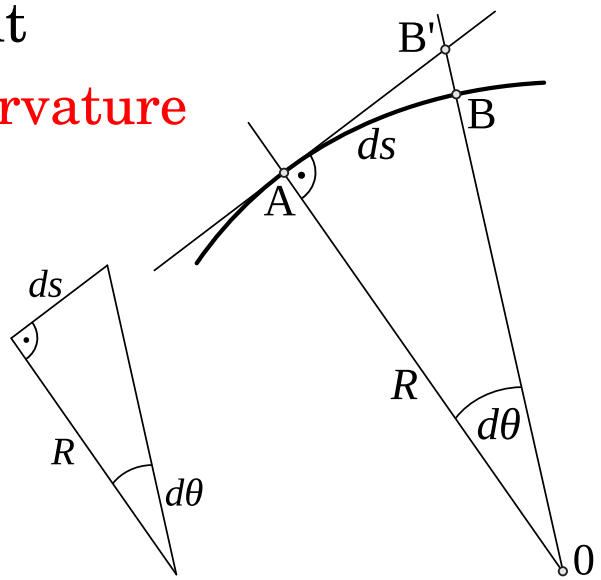


# Definition of shell

## Description of a plane curve

- let's draw a curve and the normal to that curve at point A
- let's move about $ds$ along the curve to point B and draw another normal
- the two normals intersect at point 0 called the centre of curvature
- the curvature is a measure of the rate of change of the direction of the tangent at a given point
- the length 0A is called the radius of curvature
- assuming, that $d\theta$ is very small, $ds$ goes to AB'
- thus, we can write

$$\tan d\theta \approx d\theta = \frac{ds}{R} \quad \rightarrow \quad R = \frac{ds}{d\theta}$$



# Definition of shell

## Curvature

- reciprocal of the radius is called the curvature at point and denoted with $\kappa$

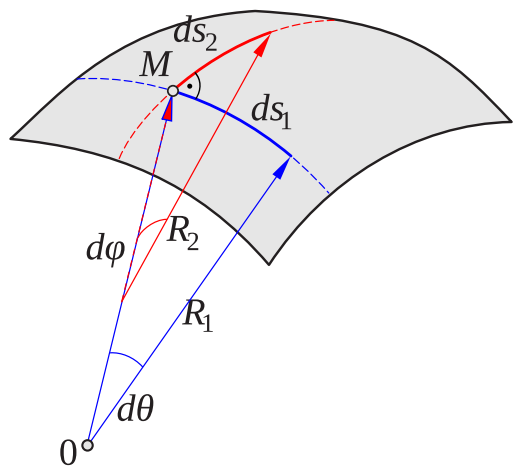
$$\kappa = \frac{d\theta}{ds}$$

- the curvature is positive when the centre of curvature lies on the negative side of the curve
- when we use the same description of the curvature in the perpendicular direction, we can describe the curvature of a shell at a given point

$$\kappa_1 = \frac{d\theta}{ds_1}; \quad \kappa_2 = \frac{d\varphi}{ds_2}$$

- and the radii

$$R_1 = \frac{ds_1}{d\theta}; \quad R_2 = \frac{ds_2}{d\varphi}$$



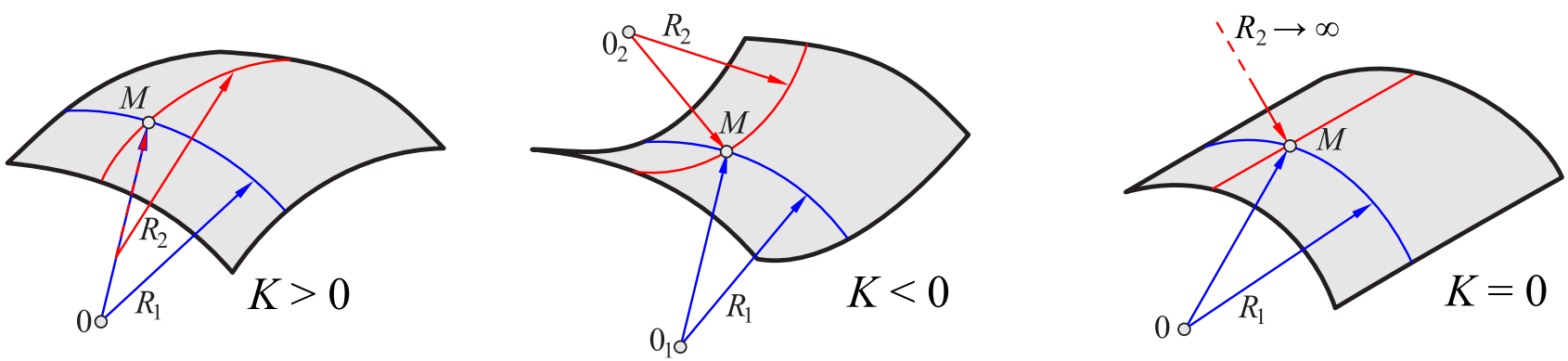
# Curvature of shells

## Gaussian curvature

- to tell something about the curvature of the shell we have to combine two curvatures defined in two directions
- the concept which help to classify shell is Gaussian curvature $K$ defined as

$$K = \frac{1}{R_1 R_2}$$

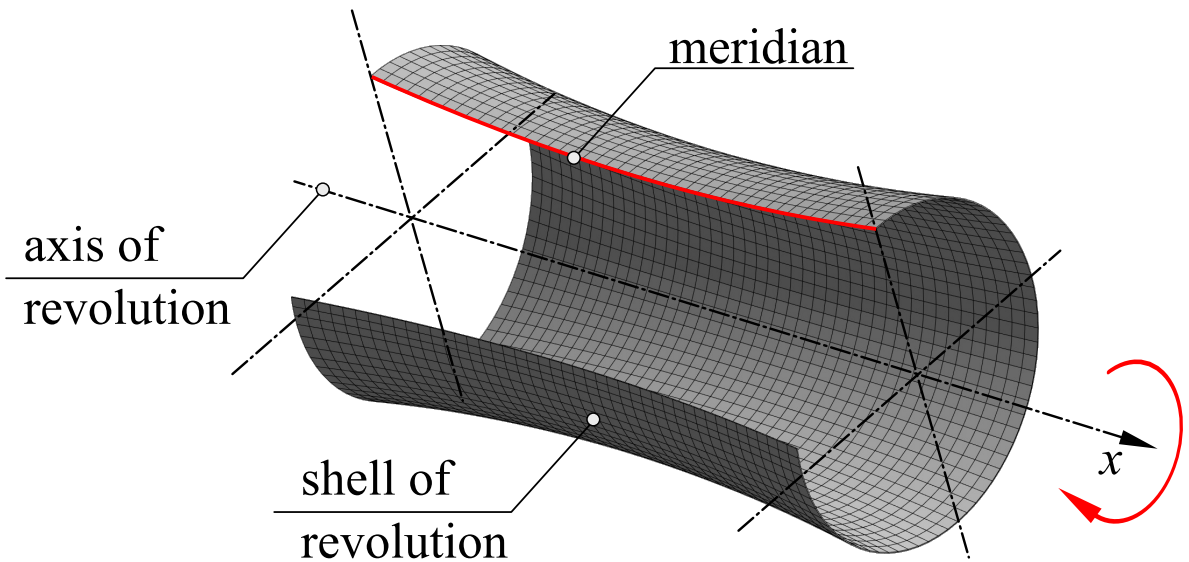
- depending on the geometry one can distinguish shells with positive, negative or zero Gaussian curvature



# Curvature of shells

## Shells of revolution

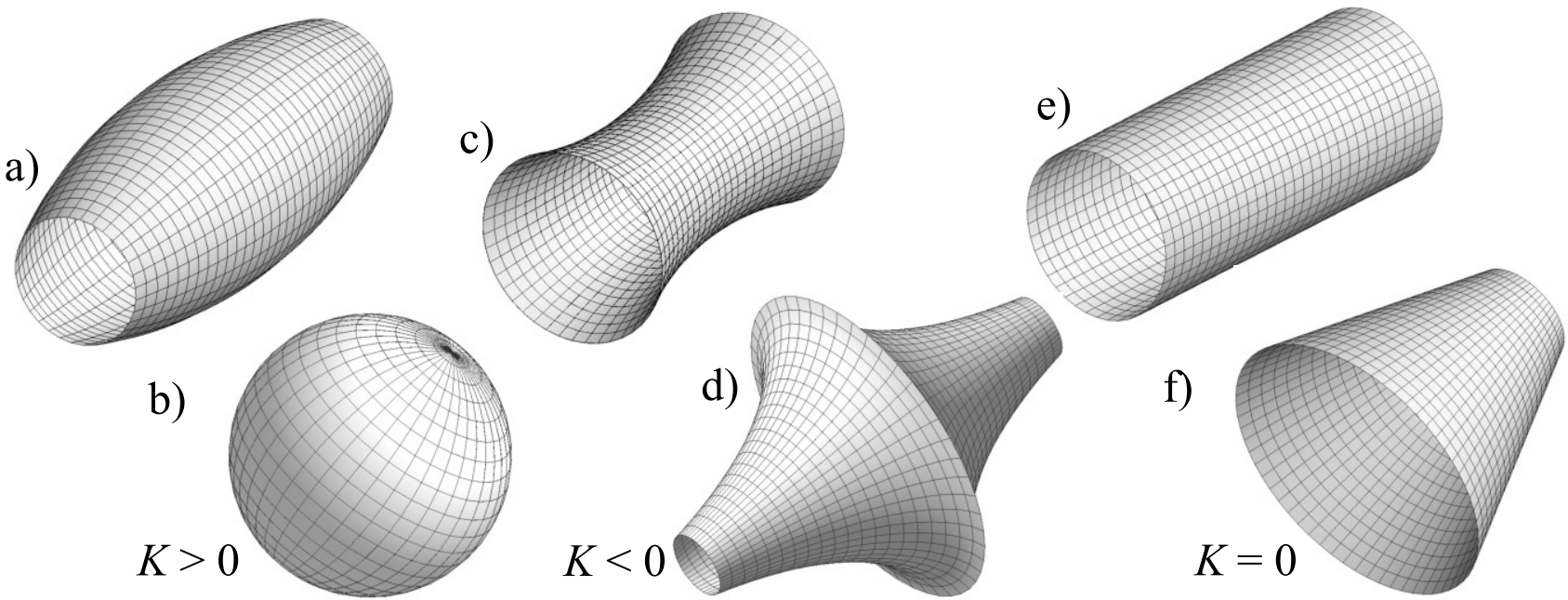
- the most popular shape of shell is a shell of revolution (SoR)
- they are obtained by revolving the curve, the meridian, around the axis of revolution
- in such shells, the meridional radius of curvature is defined by the shape of the curve
- the circumferential radius of curvature is a circle



# Curvature of shells

## Shells of revolution

- well-known examples of shells of revolution are:
  - $K > 0$: bowed-out barrel (a); sphere (b)
  - $K < 0$: bowed-in barrel (c); pseudo-sphere (d)
  - $K = 0$: cylinder (e); cone (f)

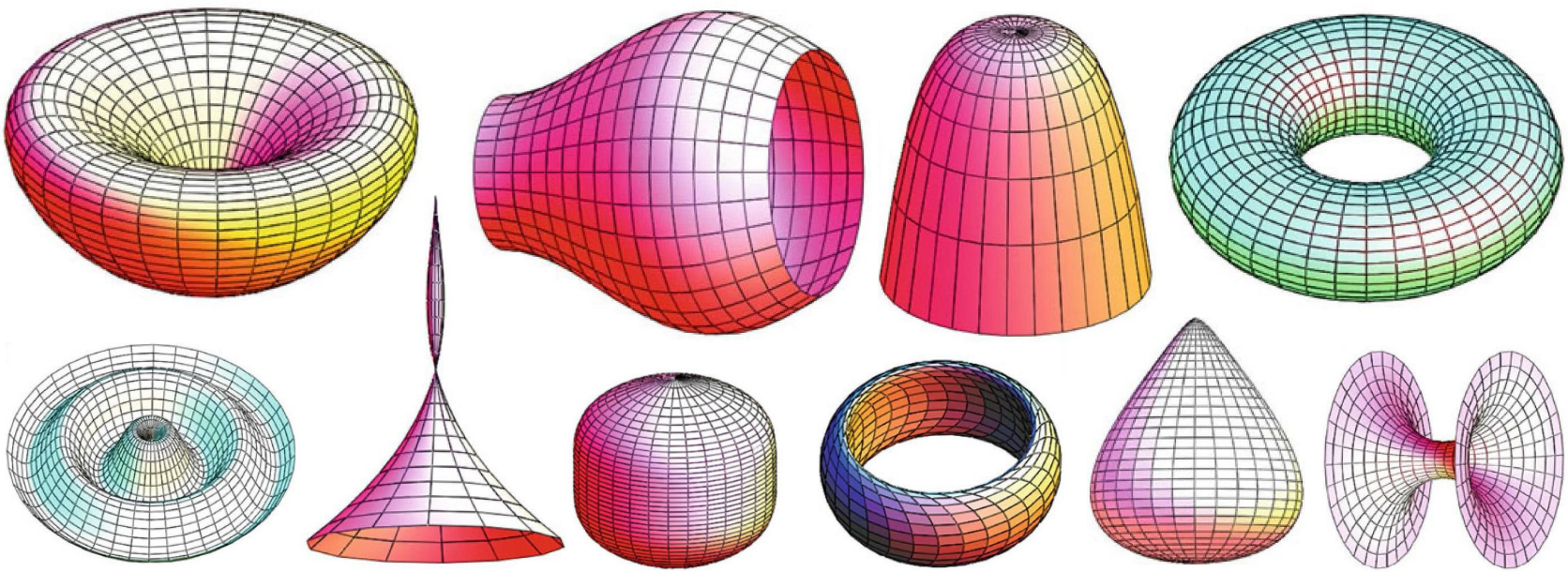


# Curvature of shells

## Shells of revolution

- it is easy to see, that there is no limitation in number of possible SoR
- many examples can be found in monograph:

  Krivoshapko SN, Ivanov VN, *Encyclopedia of Analytical Surfaces*, Springer, Cham, 2015



# Example of shell's description

## Definition of curvature

- in geometry, the curvature for the curve described with the function $y = f(x)$ is given by the formula

$$\kappa = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

- thus, the radius

$$R = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

- it can be seen that having the form of the function $y = f(x)$ we can describe the shape of the curve at any point

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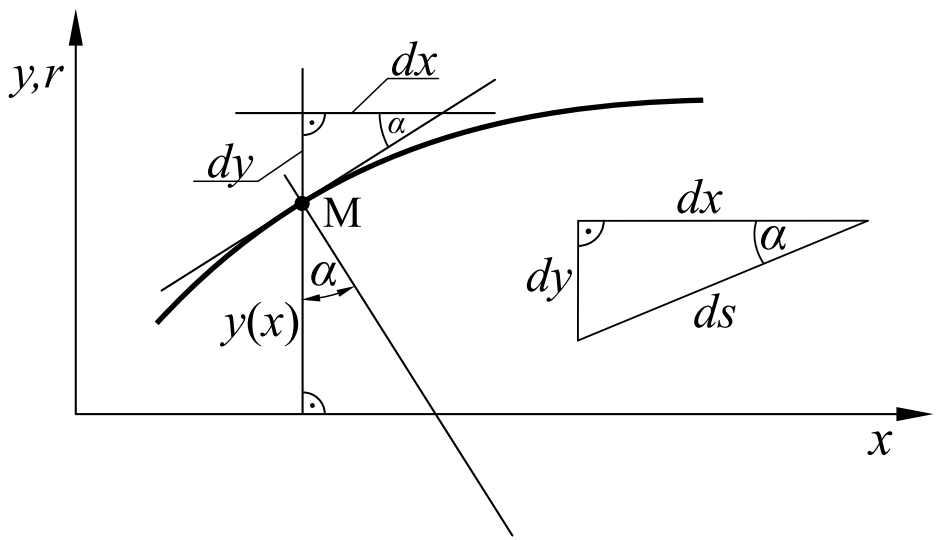



- it can be seen that having the form of the function $y = f(x)$ we can describe the shape of the curve at any point

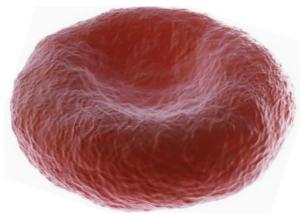
# Example of shell's description

## Cassini's ovaloidal shell

- equation of the curve: $((x-a^2)+y^2)((x+a^2)+y^2)=b^4$
- defined by points: $r_1 \times r_2 = b^2$



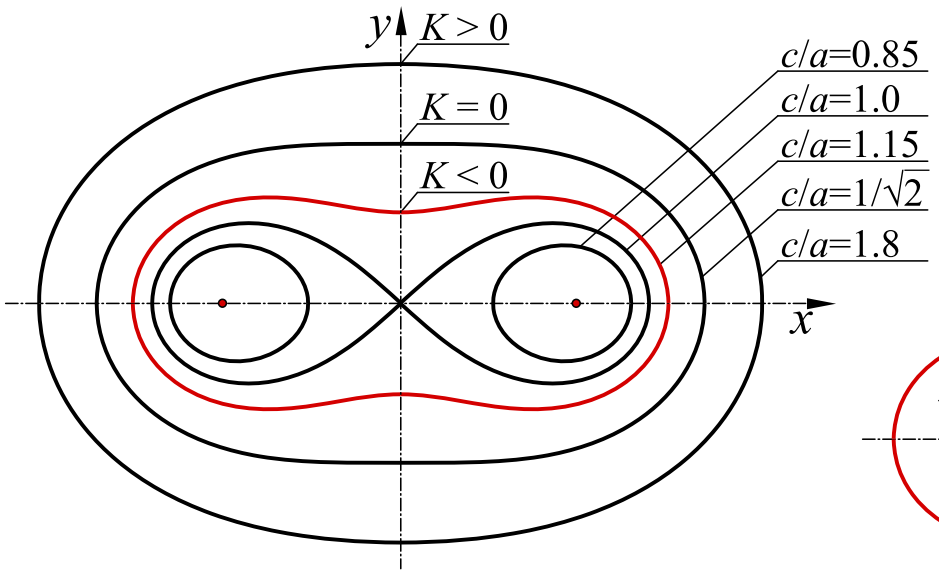
cocoon of silkworm

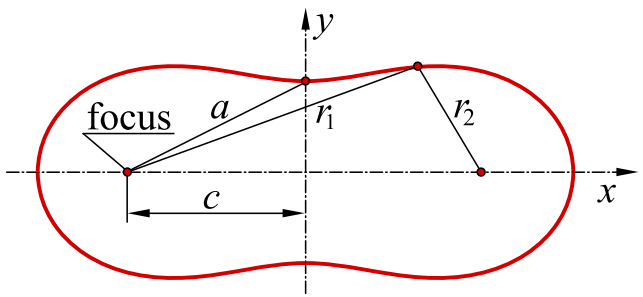


red blood cell



Giovanni Domenico Cassini



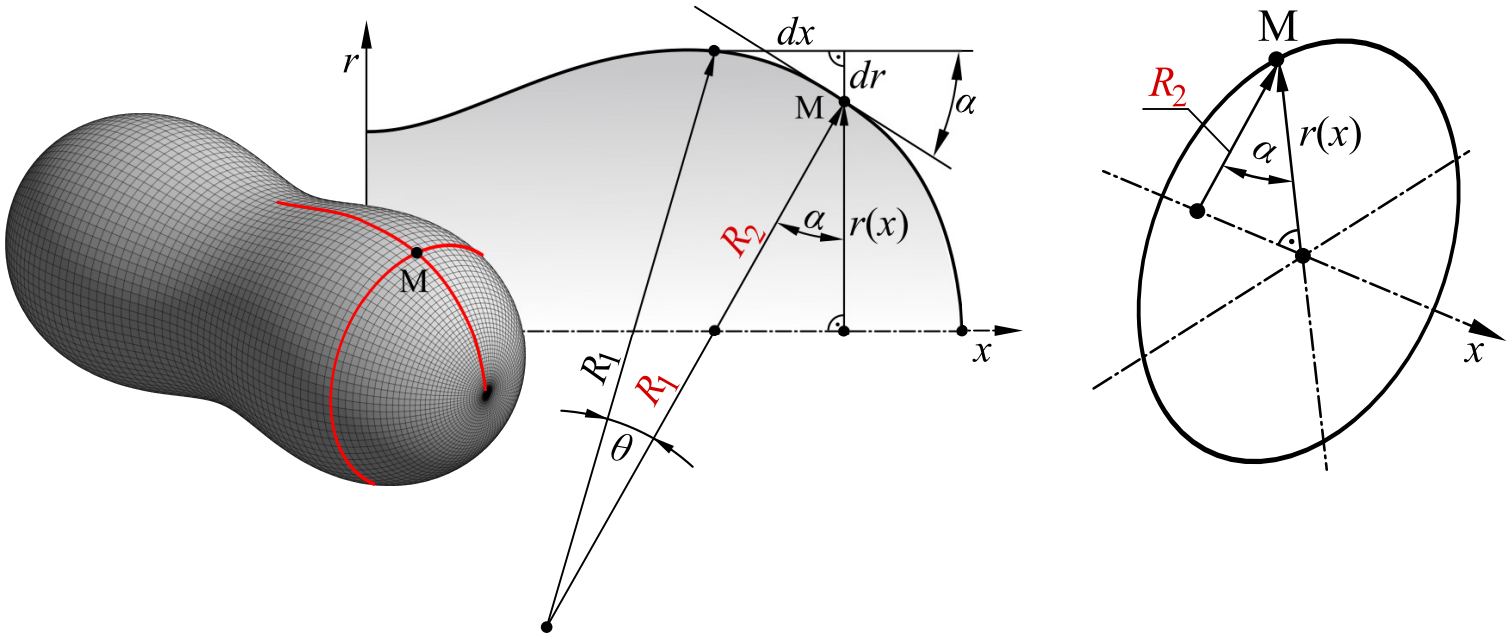



# Example of shell's description

## Cassini's ovaloidal shell

- equation of the curve:

$$y = f(x) = r(x) = \left[\sqrt{4c^2x^2 + a^4} - \left(c^2 + x^2\right)\right]^{1/2}$$



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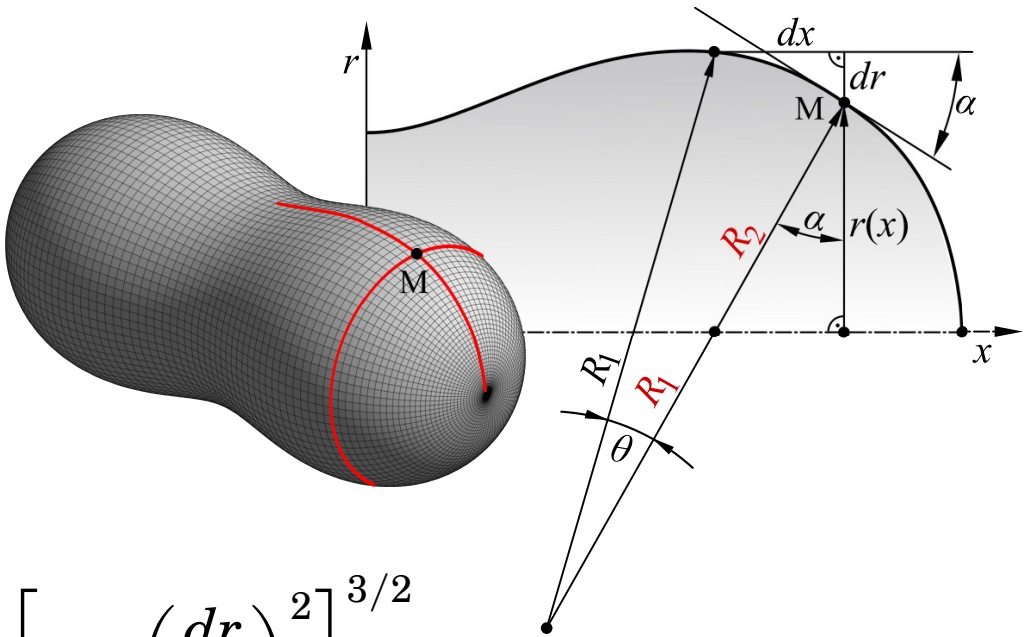


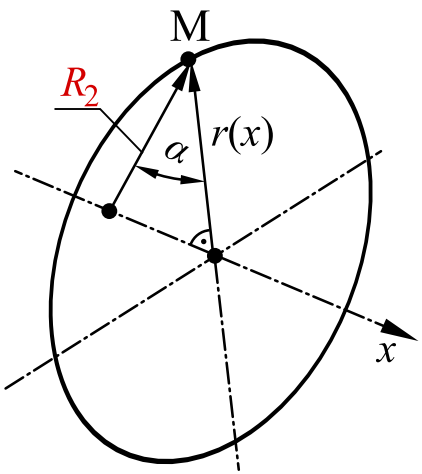


$$R_1 = -\frac{\left[1 + \left(\frac{dr}{dx}\right)^2\right]^{3/2}}{\frac{d^2r}{dx^2}};$$

# Example of shell's description

## Cassini's ovaloidal shell

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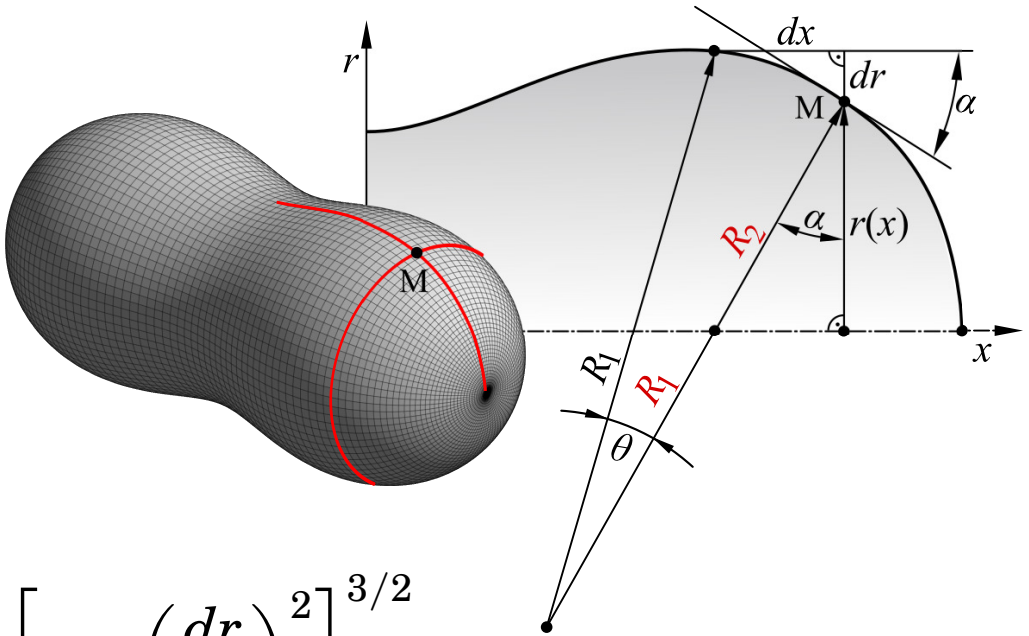
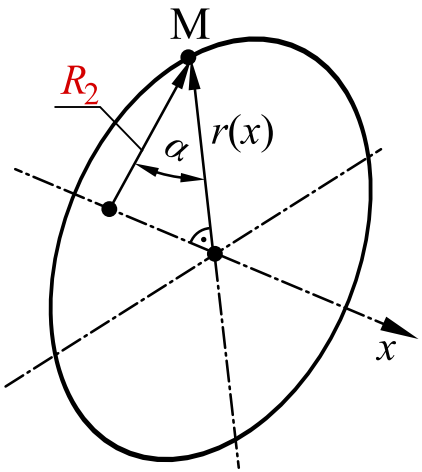



$$R_1 = -\frac{\left[1 + \left(\frac{dr}{dx}\right)^2\right]^{3/2}}{\frac{d^2r}{dx^2}}; \qquad R_2 = \left[1 + \left(\frac{dr}{dx}\right)^2\right]^{1/2} r(x)$$

# Example of shell's description

## Cassini's ovaloidal shell

- the formulas for $R_1$ and $R_2$ are valid for any shell of revolution
- to determine the geometry of such shell one has to know
  - the form of a function: $y = r(x)$
  - its first derivative: $\dfrac{dr}{dx}$
  - its second derivative: $\dfrac{d^2r}{dx^2}$
- for simple functions derivatives can be expressed in a simple functions, followed by the form of the radii
- if the function describing the meridian is more complex, a convenient way is to use some computer system

# Example of shell's description

## Cassini's ovaloidal shell – calculations

- let's assume the curve of the form

$$r(x) = \left[\sqrt{4c^2x^2 + a^4} - \left(c^2 + x^2\right)\right]^{1/2}$$

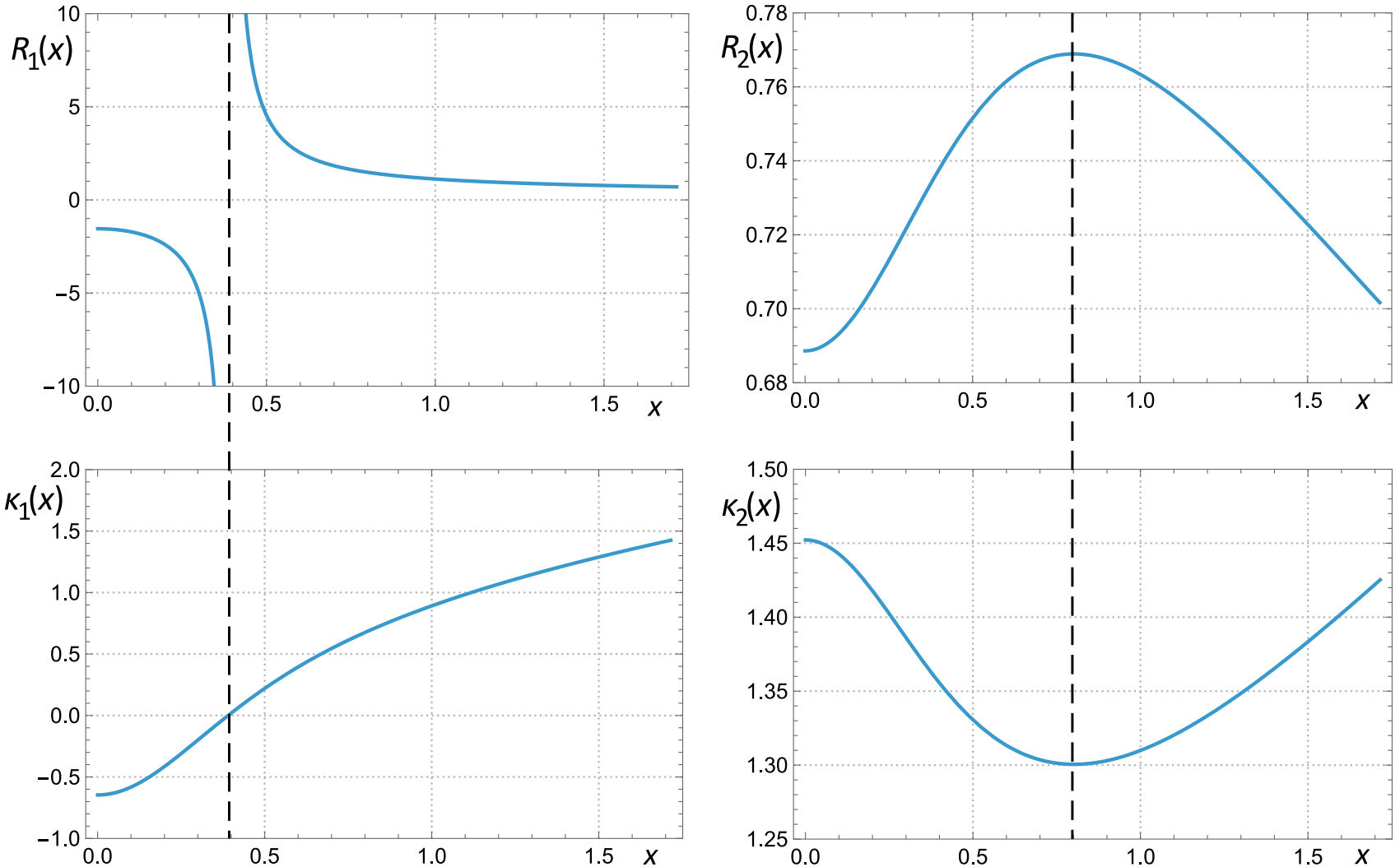
  where $a = 1.3071$ m and $c = 1.1110$ m
- main radii of curvature are given by

$$R_1(x) = \frac{(1 + r'(x)^2)^{3/2}}{r''(x)} \qquad R_2(x) = (1 + r'(x)^2)^{1/2} r(x)$$

- prepare the plots of:
  - the shape of the function
  - the radii of curvature
  - the curvatures

# Example of shell's description

## Cassini's ovaloidal shell – plots



# Curvature in practice

- the concept of curvature is important in the process of designing of aesthetic parts of motorbikes and cars
- only smooth curvature ensures a good-looking light reflections



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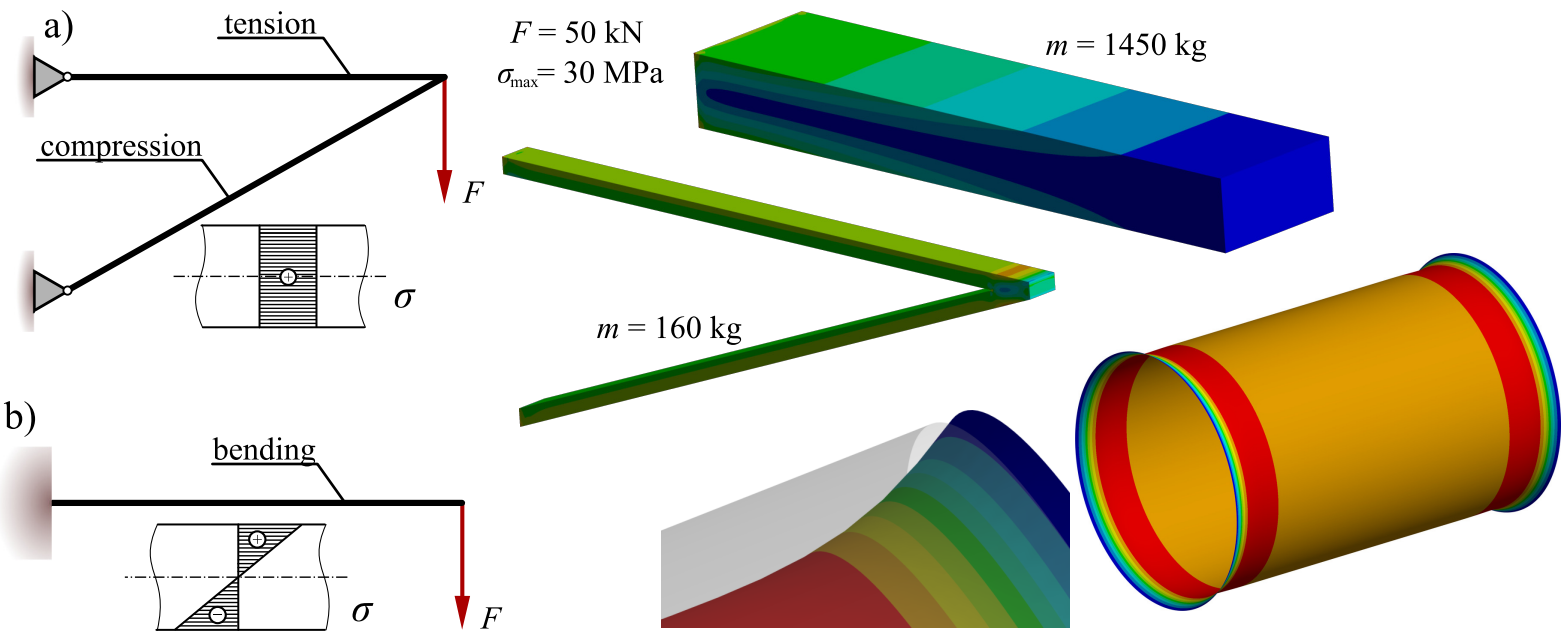
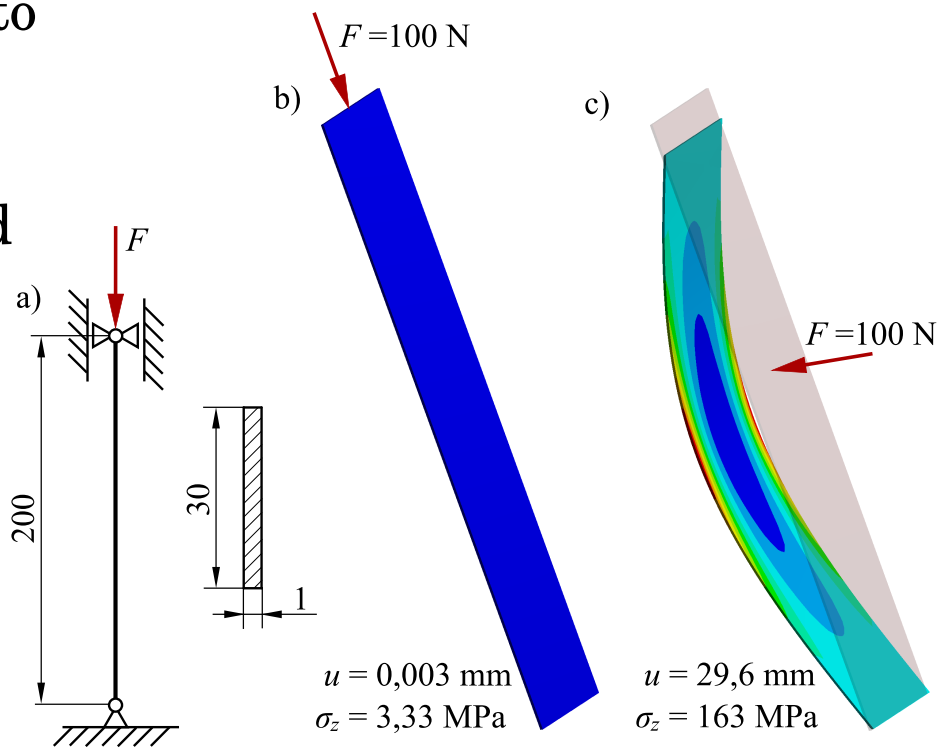
# Load carrying by membrane forces

- there are two main modes to bare the load – membrane forces and bending
- membrane forces – tension and compression – is much more effective mode
- in shells both modes are present; however the bending should be diminished



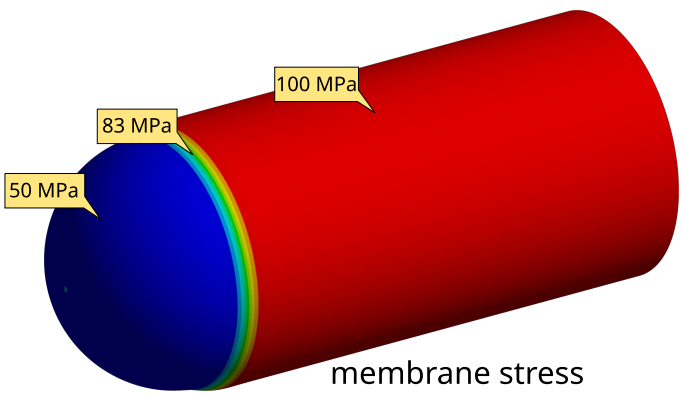
# Load carrying by membrane forces

- slim and thin-walled structures bare the load through the membrane forces
- deformations corresponding to this forces are very small
- but the deformation due to transverse forces are big
- thus slim and thin-walled structures have large membrane stiffness and small bending stiffness

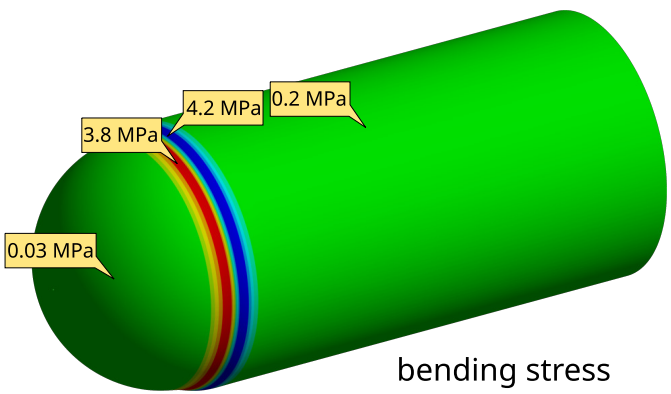


# Load carrying mechanism in shells

- the shell transfers the transverse load mainly through tensile and compressive forces – membrane forces
- they are equally distributed through the thickness of the shell; such state of stress is called the membrane state of stress
- the advantage of membrane state is that
  - it allows to take full advantages of the mechanical properties of the material – in pure tension, all fibres on the cross section are equally strained
  - for a given value of load the membrane stresses are always smaller than the bending stresses



membrane stress

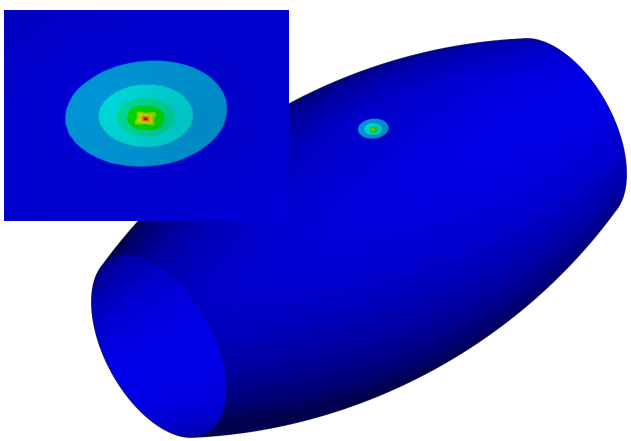


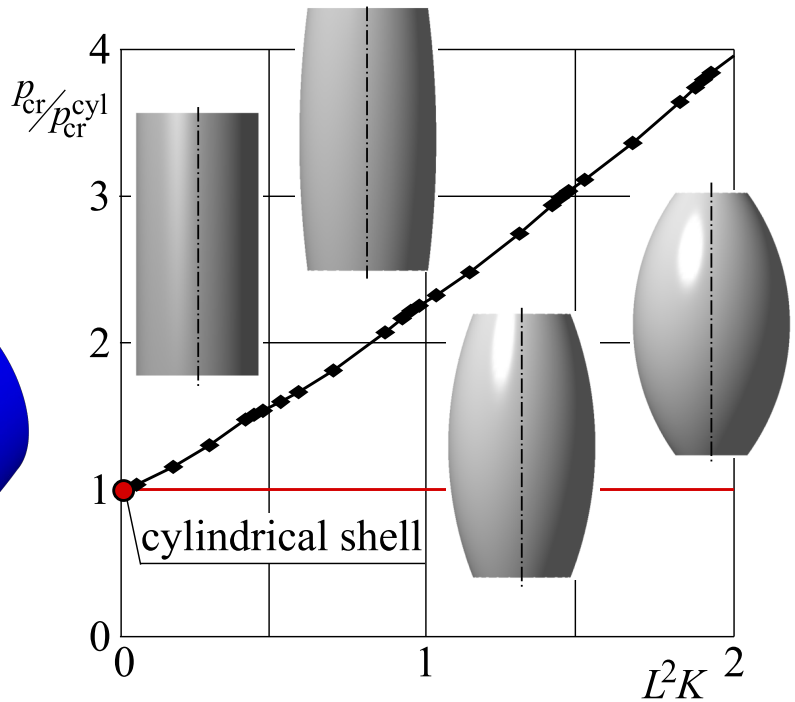
bending stress

# Load carrying mechanism in shells

## Source of effectiveness

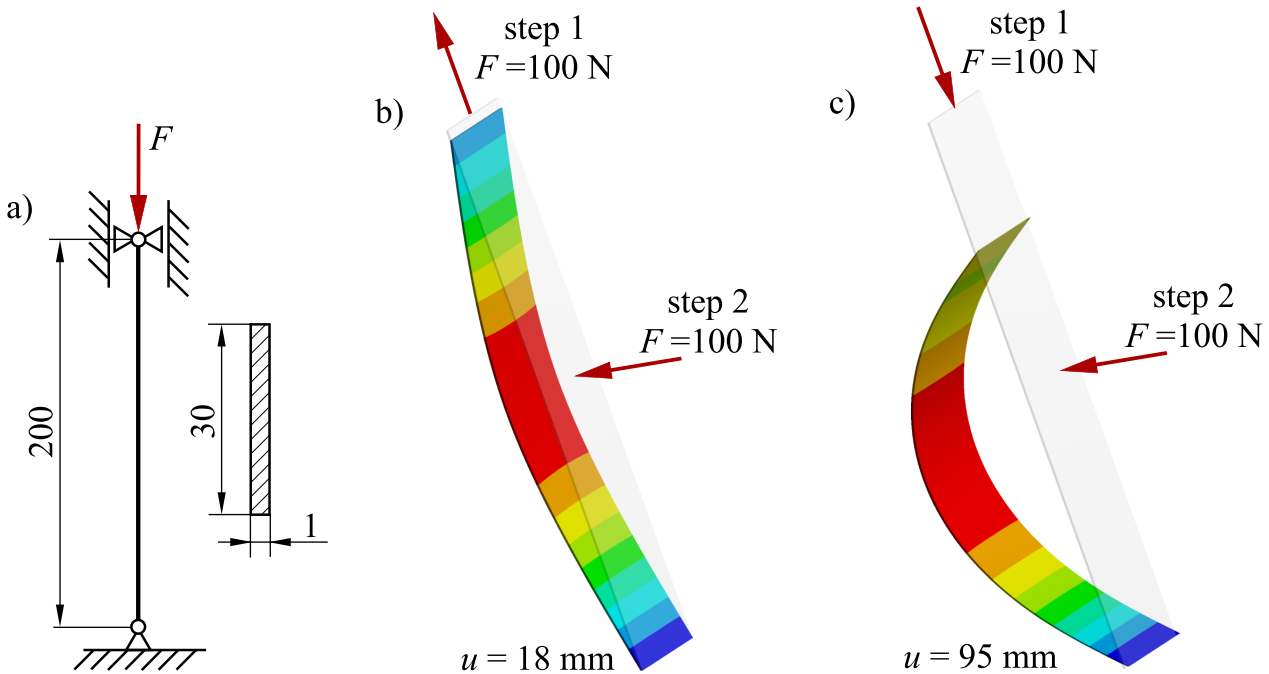
- the effectiveness of the shell as a structural element is related with they curvature and thin-walled
- thanks to curvature, the shell gains spatial rigidity, and the load is distributed to membrane forces
- thanks to thin-walled character
  - the shell gains lightness
  - locally occurring bending stresses quickly disappear





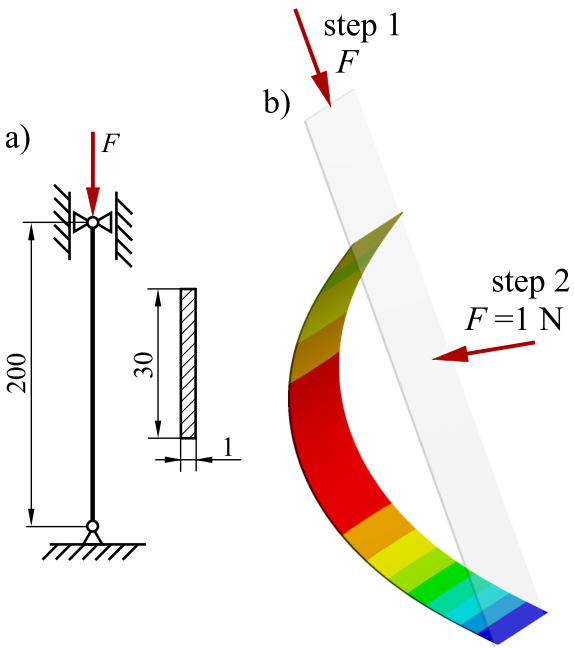
# Influence of membrane forces on stiffness

- membrane forces influences the bending stiffness of the structure in two ways
  - tensile forces increase the stiffness (*stress stiffening*) (b)
  - compression forces decrease the stiffness (c)



# Disadvantages of compressive membrane forces

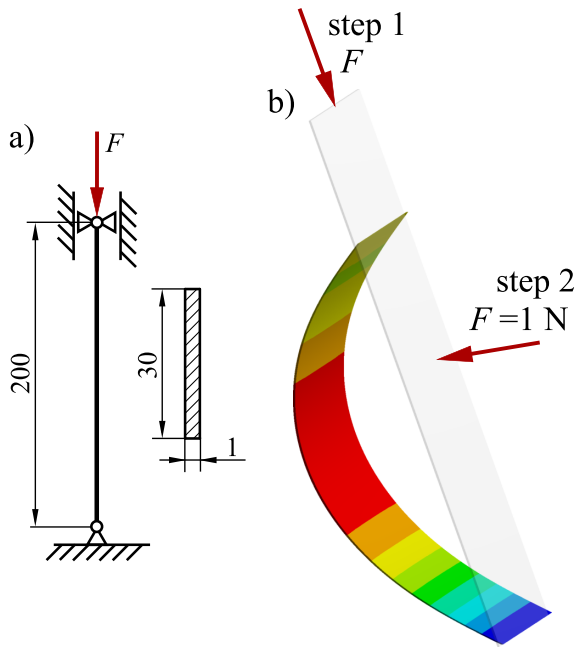
## Structure with zero stiffness

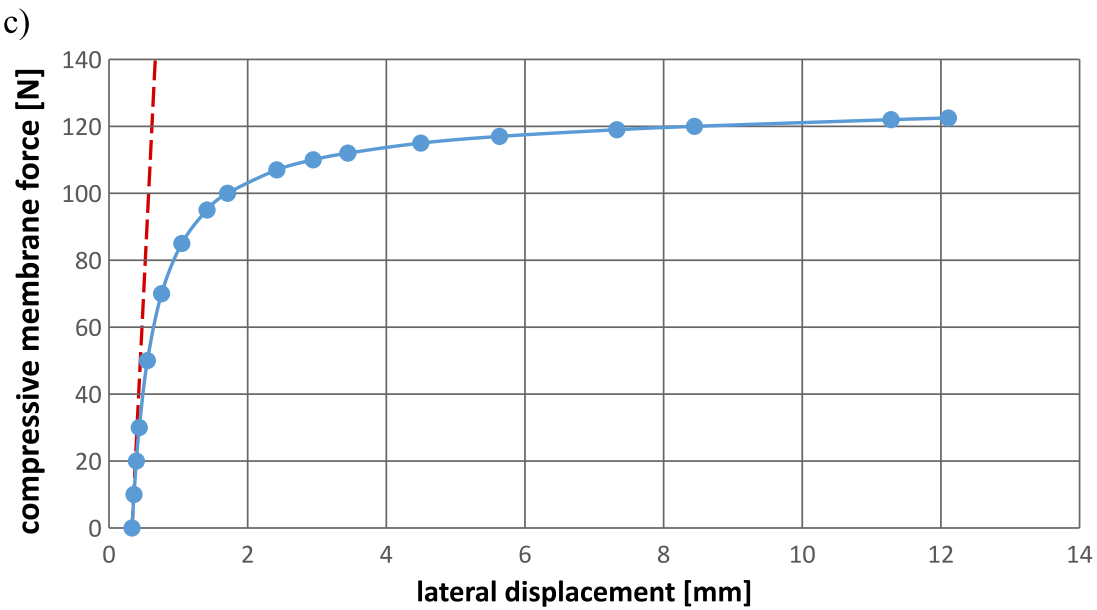


# Disadvantages of compressive membrane forces

## Structure with zero stiffness

- increasing compressive membrane forces decrease the stiffness of the structure
- at some value of compressive load the displacement goes to infinity, thus the stiffness of the structure goes to zero



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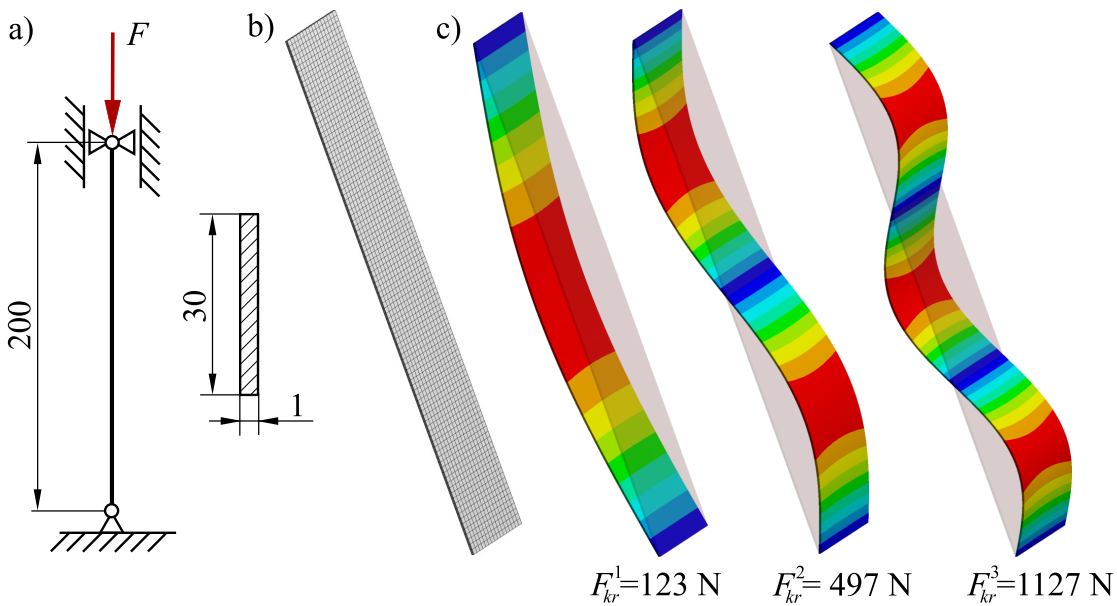
# Structure with zero stiffness – buckling

- if the stiffness of the structure drops to zero, any small load (disturbance) may lead to loss of its stability in a sudden process called buckling

## Buckling

a physical process involving a sudden change of the shape of a structure



# Energy and buckling

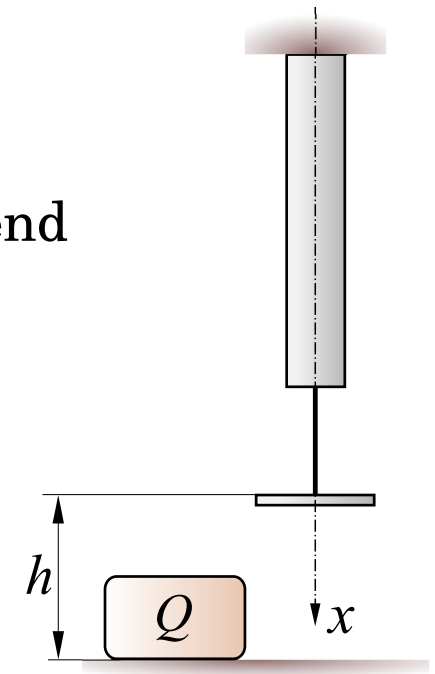
- the phenomenon of the loss of stability can be explained with the use of energy approach
- to do this we will use
  - the work-energy theorem
  - the law of conservation of energy
- we will start by explaining how the energy of the external load is transformed into the energy of elastic deformation

# Energy and buckling

## Equivalence of work and energy

- as an example let's consider the vertical bar fixed at the upper end, and a rigid disc fixed at the other
- next to the bar there is a weight $Q$, which will be placed on the disc
- after being elevated, the weight gains energy and, when dropped onto the disc, it will descend due to the elongation of the bar

$$E_p^Q = mgh = U_{total}$$

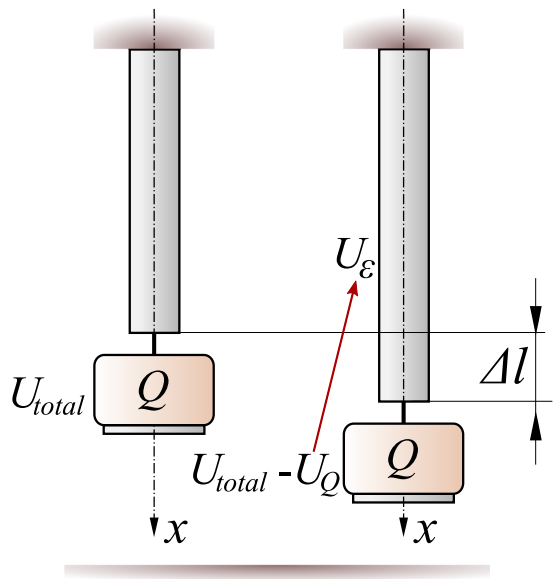


# Energy and buckling

## Equivalence of work and energy

- by lowering about $\Delta l$, the weight's total energy decreases by $U_Q$
- this energy is transformed into energy of elastic deformation $U_\varepsilon$
- both energies can be expressed by the work – energy of the weight as the work of the external forces $(W_e)$, and the energy of elastic deformation as a work of internal forces $(-W_i)$
- which is

$$U_Q = W_e \qquad \text{and} \qquad U_\varepsilon = -W_i$$



# Energy and buckling

## Equivalence of work and energy

- according to the law of conservation of energy for elastic systems

$$U_\varepsilon = U_Q$$

- finally, because $U_Q = W_e$

$$U_\varepsilon = W_e$$

# Energy and buckling

## Equivalence of work and energy

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- finally, because $U_Q = W_e$

$$U_\varepsilon = W_e$$

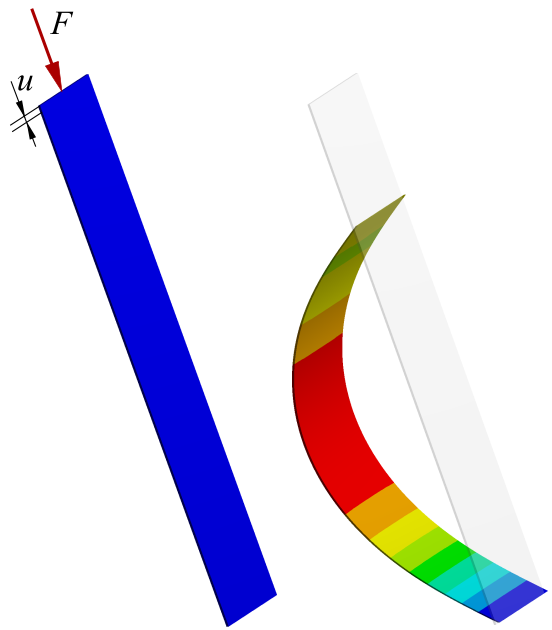
**Attention!**

- the work can not be transformed into energy
- only a different form of energy can be converted into potential energy of elastic deformation
- the work $W_e$ is only a measure of the potential energy of the external forces

# Energy and buckling

- at the moment of loss of stability, the energy of deformation of the membrane state is transformed into the energy of bending state

$$U_{\varepsilon}^{\text{memb.}} = \frac{1}{2} F u = U_{\varepsilon}^{\text{bend.}}$$



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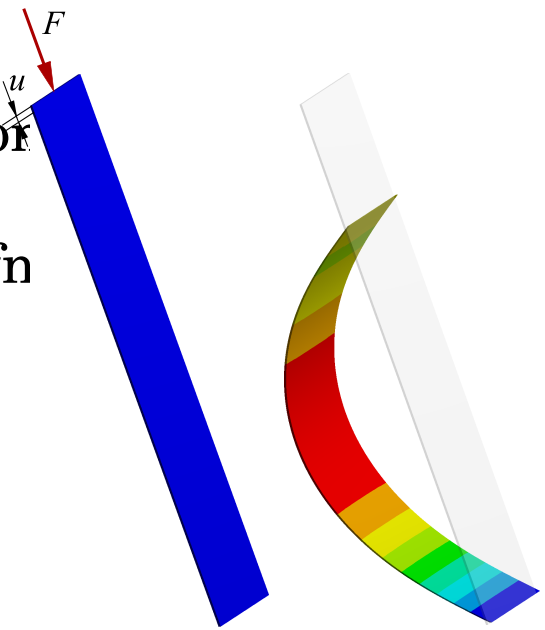

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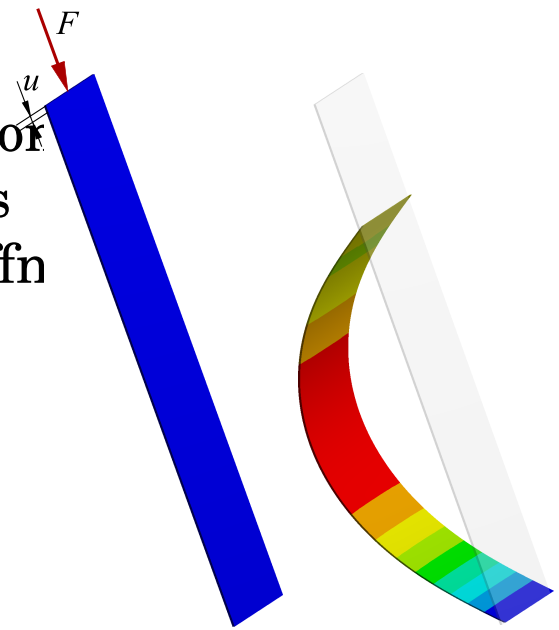
- since the amount of the energy remains on the same level, and the bending stiffness is much smaller than the membrane stiffn a large deformation is necessary to absorb the energy stored in the structure
- it usually leads to the failure of a structure



# Buckling of shell structures

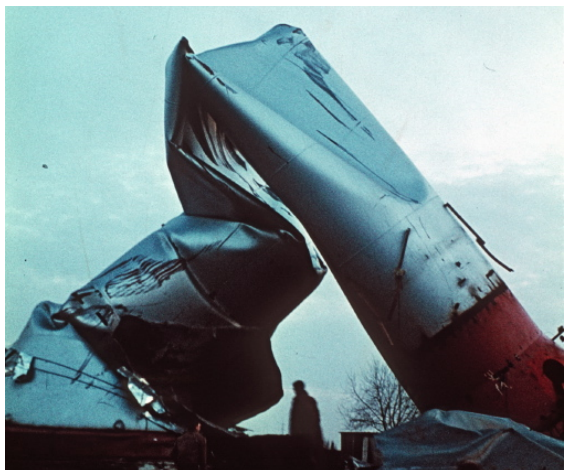
## Disastrous results of buckling



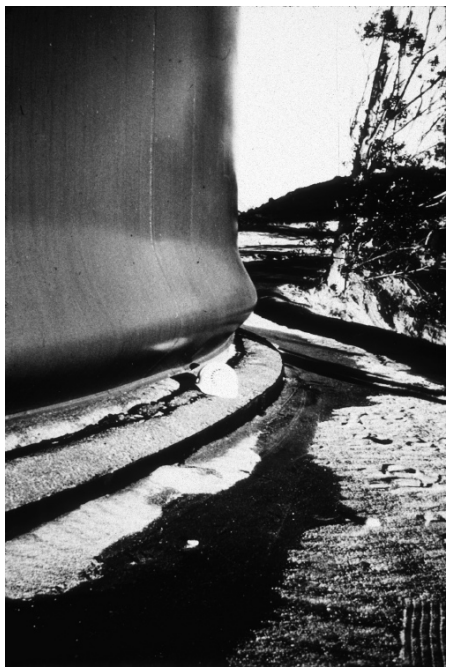
cylindrical shell - vertical tank
*vacuum pressure*



cylindrical shell - vertical tank
*liquid internal pressure*



cylindrical shell - chimney
*lateral pressure*



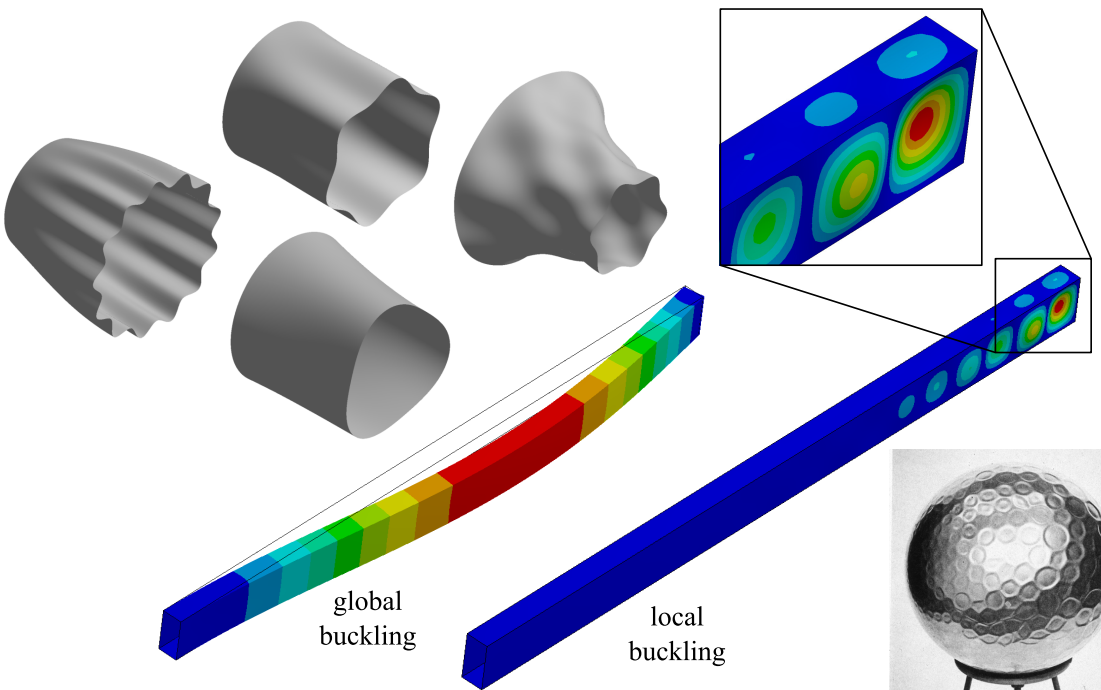
cylindrical shell - vertical tank
*liquid internal pressure*

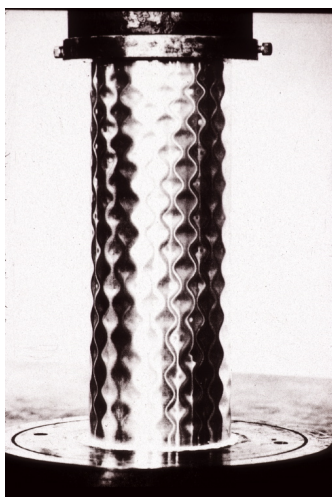
# Buckling of shell structures

## Buckling mode

the shape the structure takes after buckling

- the buckling mode has usually the form of waves
- it may have global or local character



cylindrical shell
*axial compression*

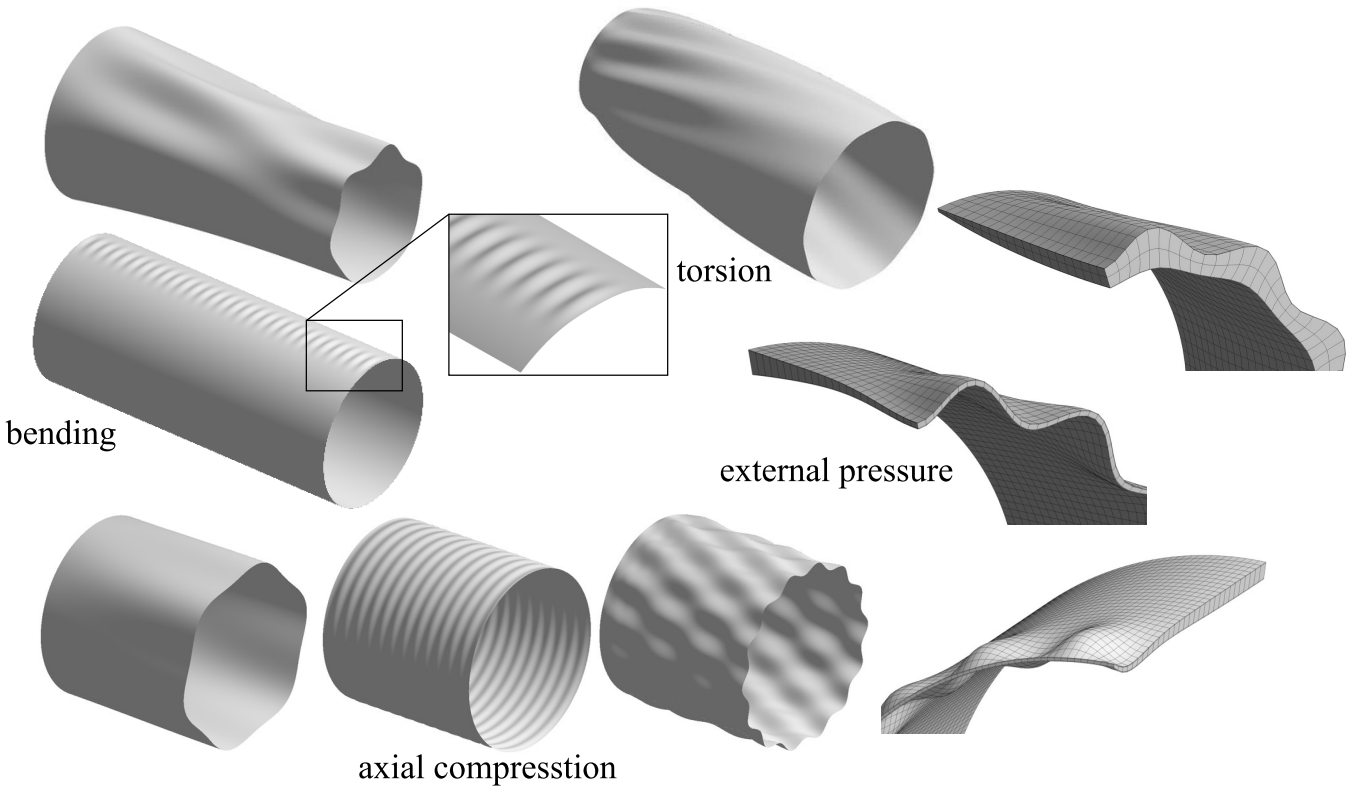


spherical shell *external pressure*

# Buckling of shell structures

The buckling mode depends on many factors:

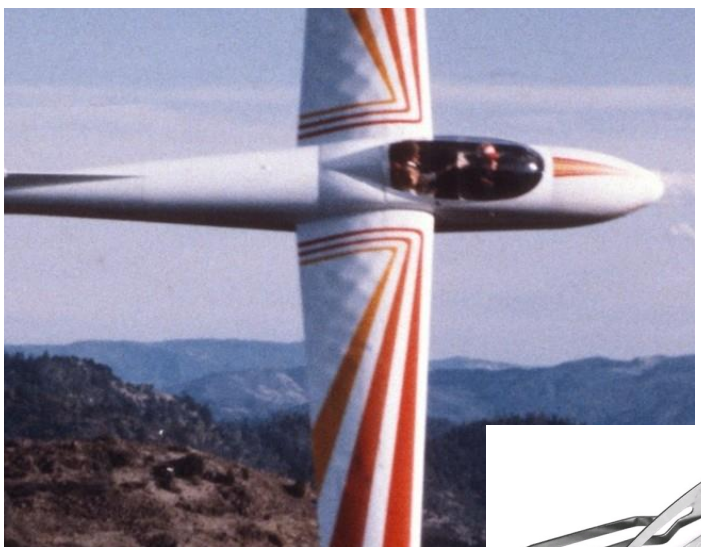
- geometrical parameters and relations between them,
- structure of the cross-section,
- boundary conditions – load and support.



# Buckling of shell structures

## Buckling in real life

- sometimes buckling is the result of normal (designed) behaviour



*glider wing*
wrinkles on compressed side



*jar lid*
airtight seal
by snap-buckling



*trousers leg*
buckling due to compression



*chair clip*
closing by
snap-buckling

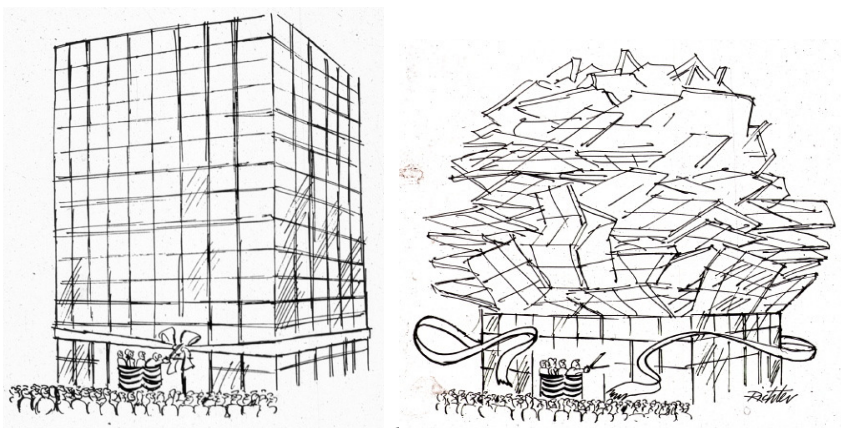


*fluttering flag*
dynamic buckling

# Buckling of shell structures

We already know, that the shell may buckle due to compressive membrane forces. But many questions arise, such as:

- Can we control the process of buckling?
- Can we influence the buckling shape or buckling load?
- Is the structure in equilibrium after buckling?
- Can the buckled structure bear the load?
- etc.



*www. shellbuckling.com*

# Buckling of shell structures

In order to answer these questions we must first:

- establish some order in terms such as:

  EQIULIBRIUM

  *buckling* ? **stability**

- discover how shells have to be analysed to obtain a full picture of their response to the load



*I am in equilibrium.*
*But am I stable?*

# Types of equilibrium

- in statics, objects in equilibrium are analysed; it is not important what is the type of equilibrium

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stability is a system property that describes how it responds to disturbances to its equilibrium state

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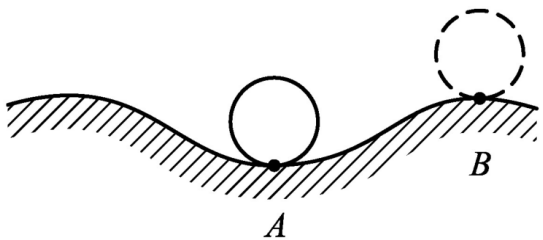
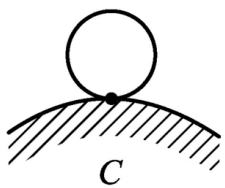
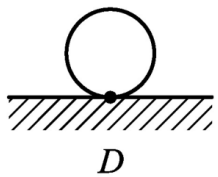


stable equilibrium



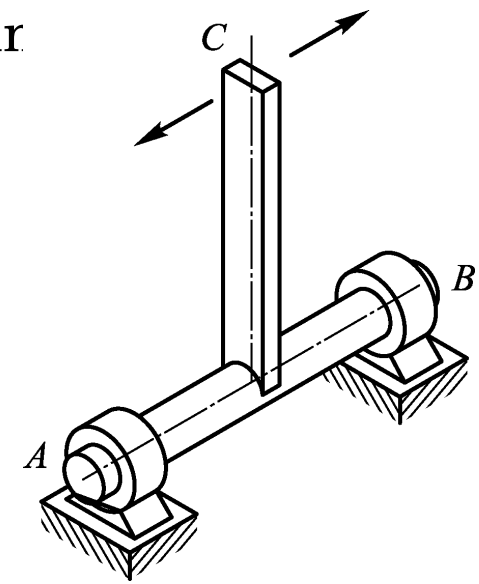
unstable equilibrium



neutral equilibrium

# Types of equilibrium

- the response of the system to a disturbance of its equilibrium state depends on
  - type of structural element (column, plate, shell)
  - geometrical dimensions
  - type of load and support conditions
  - initial geometrical imperfections
- thus, stability analysis must be performed ir given system, taking into account different types of disturbance.

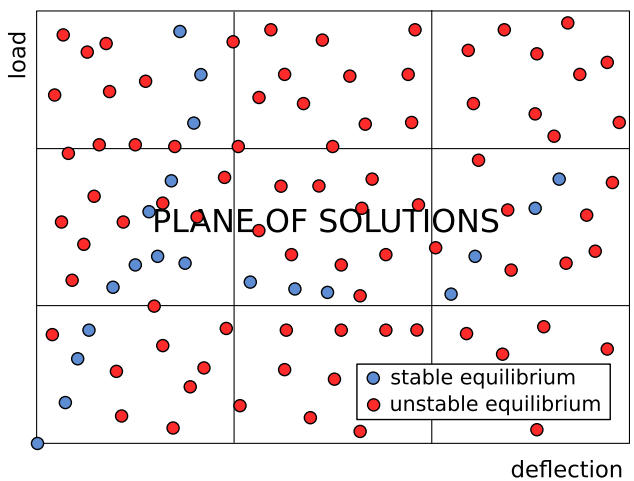


# Equilibrium path

- to fully describe the behaviour of the structure, one has to follow the load-displacement history
- this means that, for an incrementally increasing load, the position of the selected point has to be determined
- this load history is called equilibrium path

## Equilibrium path

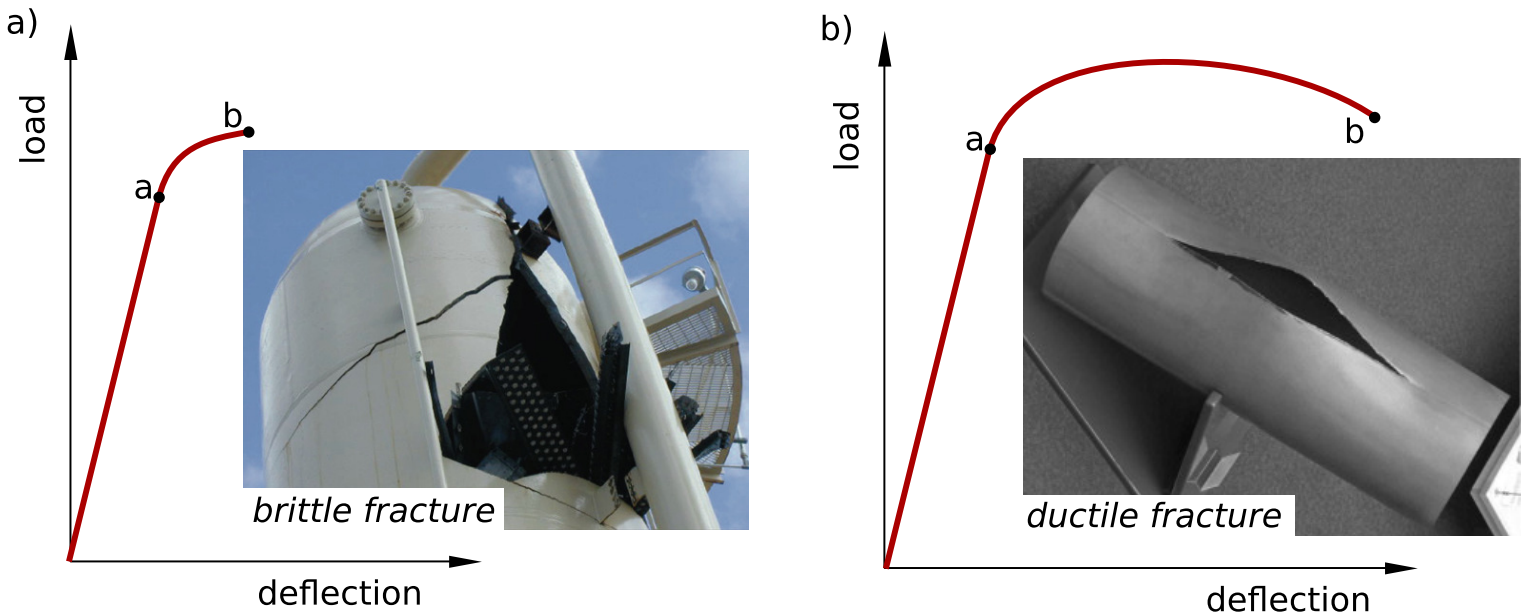
relation between the load and the displacement of a particular point of the structure



# Equilibrium path

## Behaviour of typical structures

- two typical load-displacement histories are related to brittle fracture and ductile fracture
- in both cases, the equilibrium path is smooth and finishes with the failure



# Equilibrium path

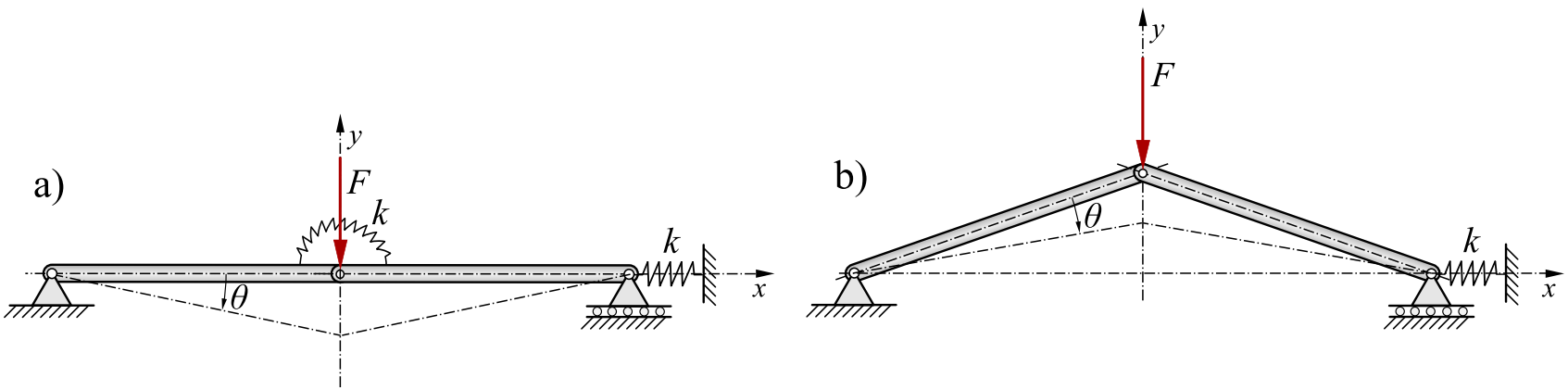
## Non-linear stiffening and unstiffening

As we already know

- tensile membrane forces stiffen the structure
- compressive membrane forces reduce the stiffness of the structure

Let's analyse the behaviour of two simple models that reflect two types of elastic structures

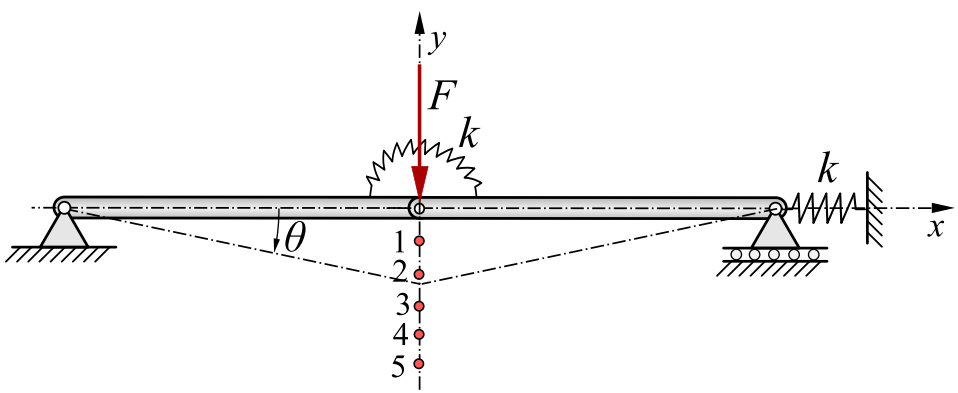
- characterised by non-linear stiffening
- characterised by non-linear unstiffening

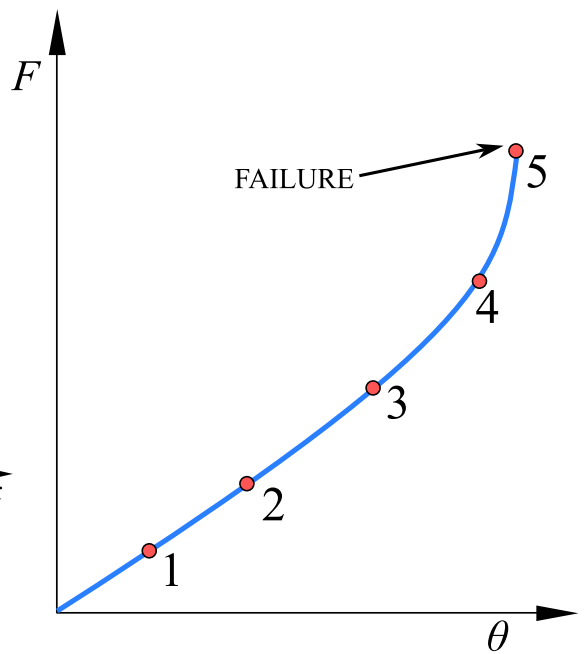


# Equilibrium path

## Non-linear stiffening behaviour

- in the case of non-linear stiffening behaviour, the stiffness of the structure is constant at the beginning and increases at higher deformations
- the failure appears due to fracture of plasticisation

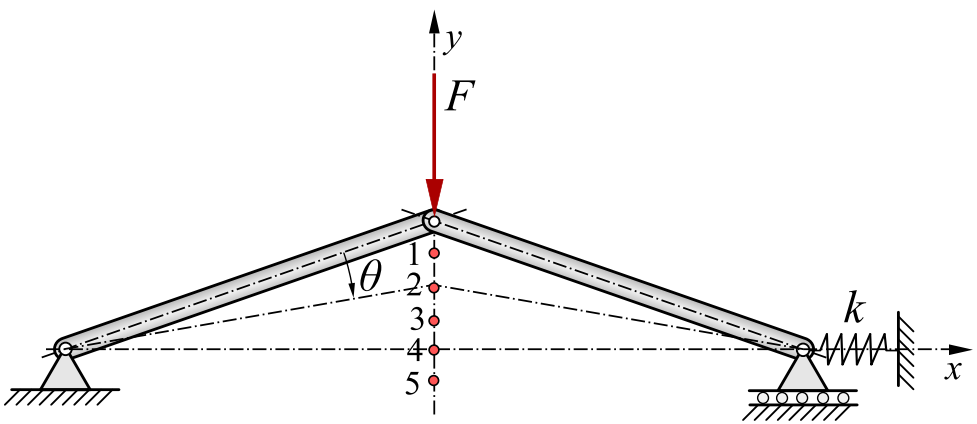


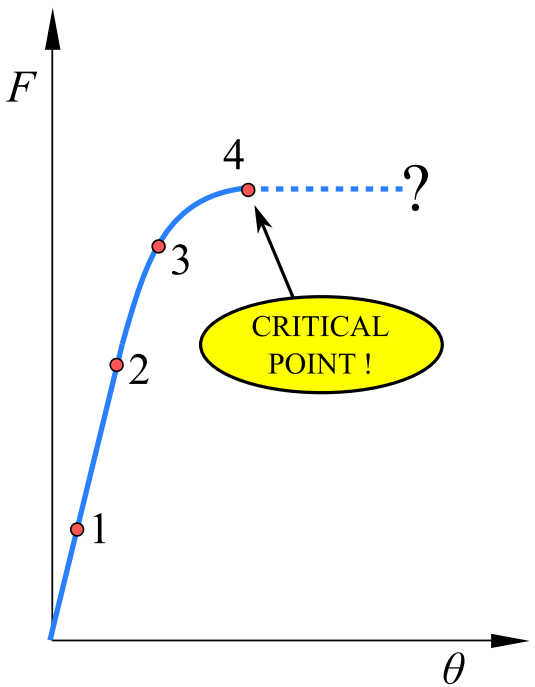



# Equilibrium path

## Non-linear unstiffening behaviour

- in the case of non-linear unstiffening behaviour, the stiffness of the structure is constant at the beginning and decreases at higher deformations
- the failure appears due to...?

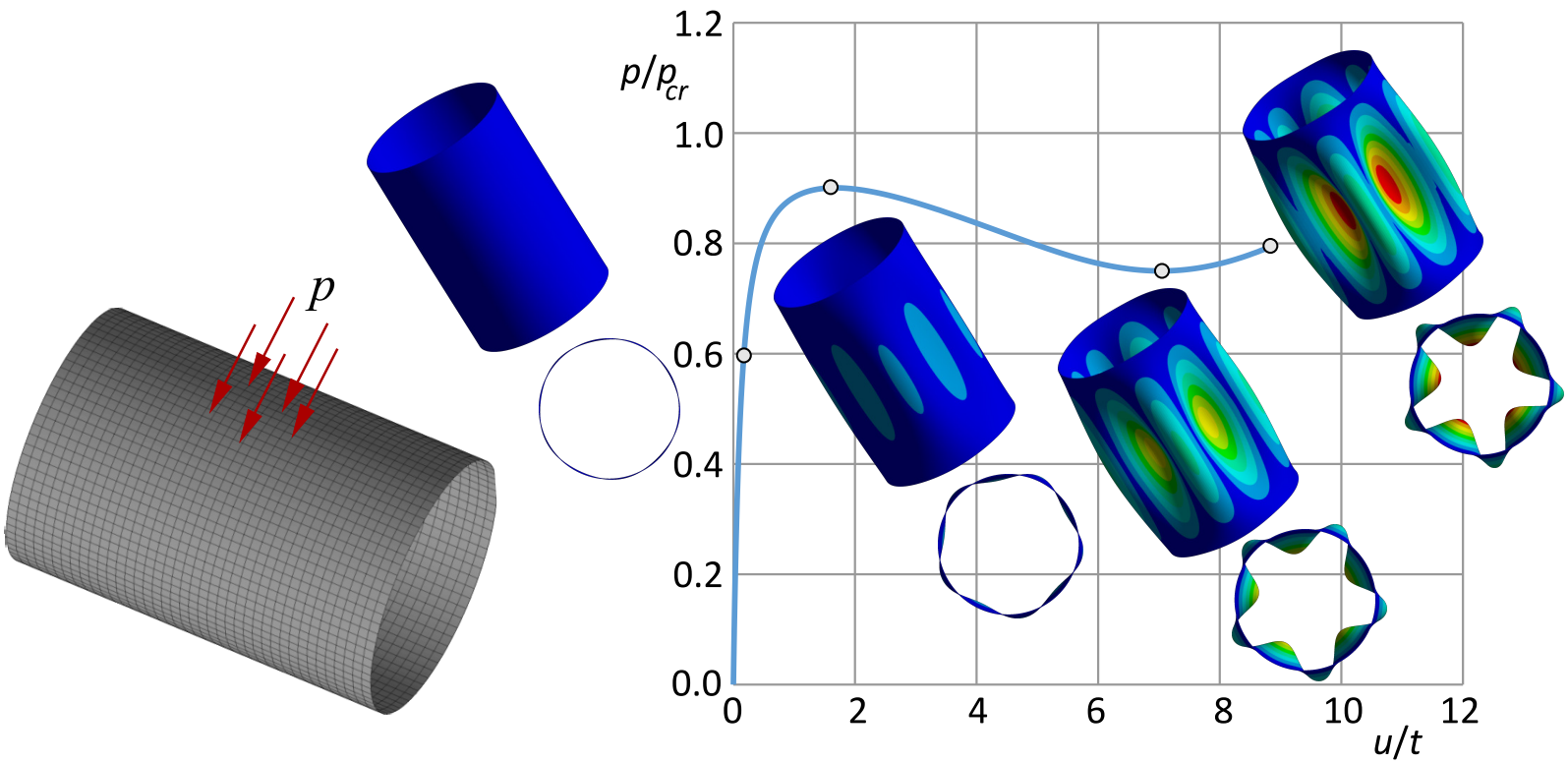


# Equilibrium path

## Cylindrical tank under external pressure

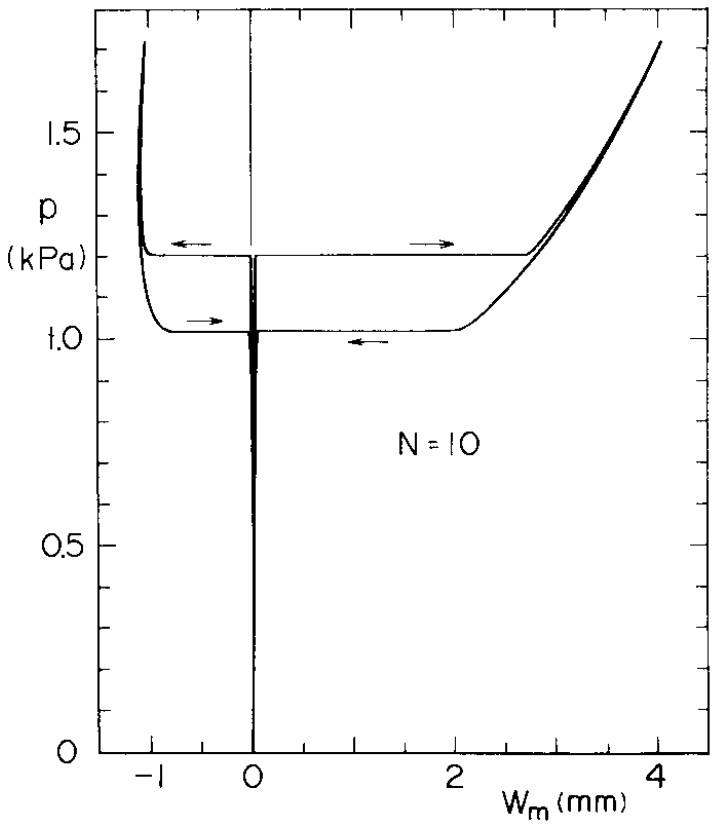
- let's analyse the example of cylindrical shell loaded with external pressure
- the results of numerical analysis is shown below

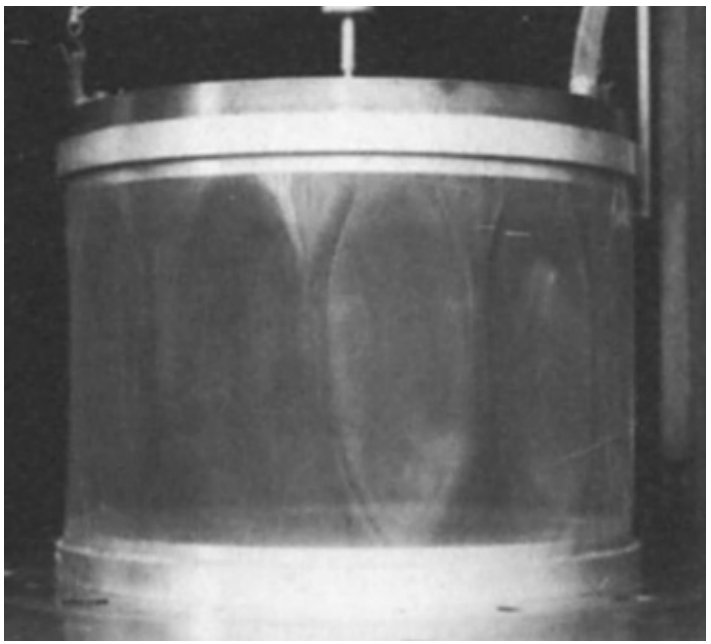


# Equilibrium path

## Cylindrical tank under external pressure

- experiments conducted by Yamaki reveal actual behaviour of cylindrical shells under external pressure



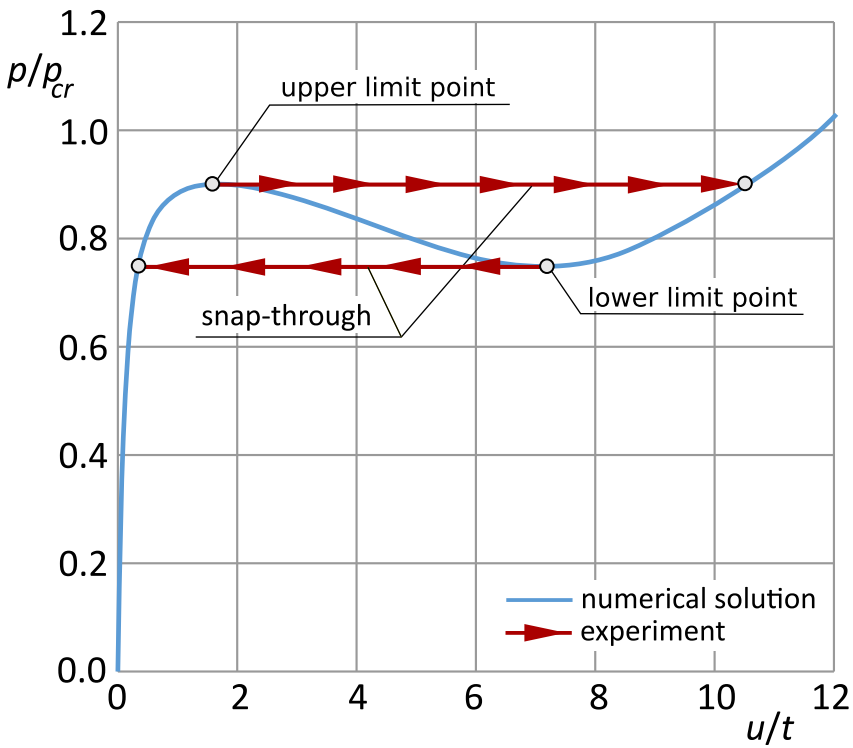



Yamaki N. *Elastic stability of circular cylindrical shells*, Elsevier, 1984

# Equilibrium path

## Cylindrical tank under external pressure

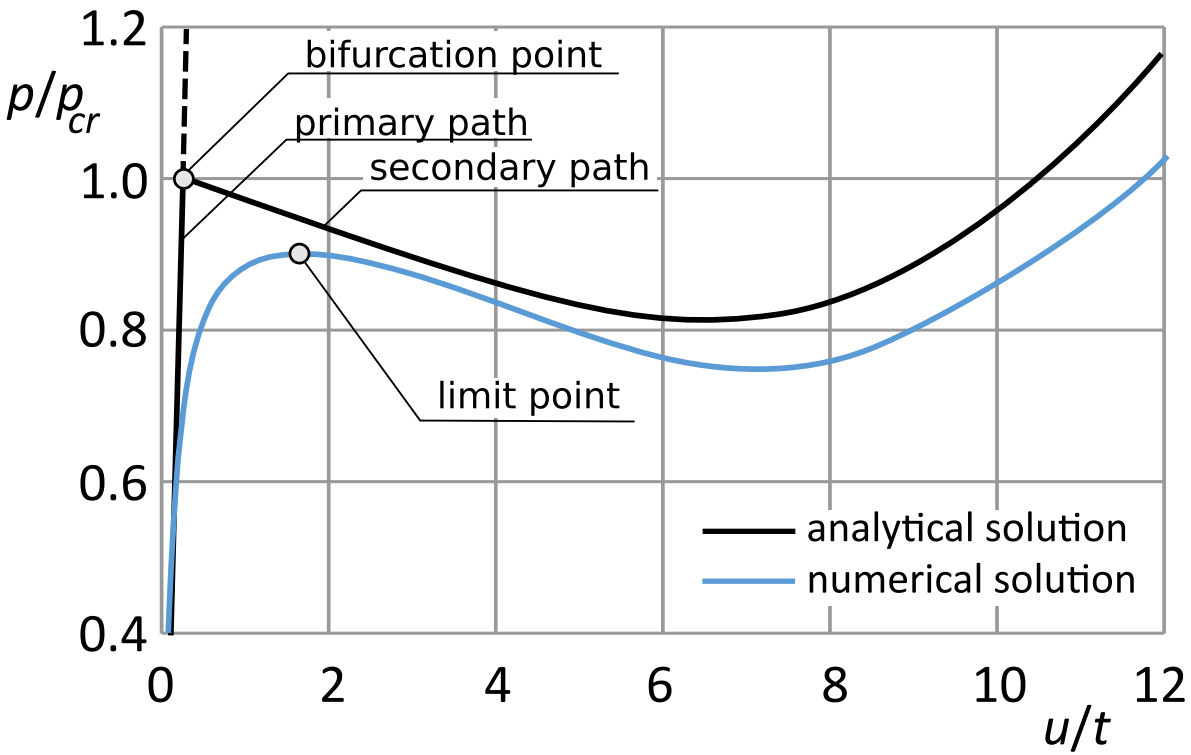
- the path obtained numerically is the result of the procedure used – it allows to control the value of the load
- in experiments or in real-life situations, load is constantly growing



# Equilibrium path

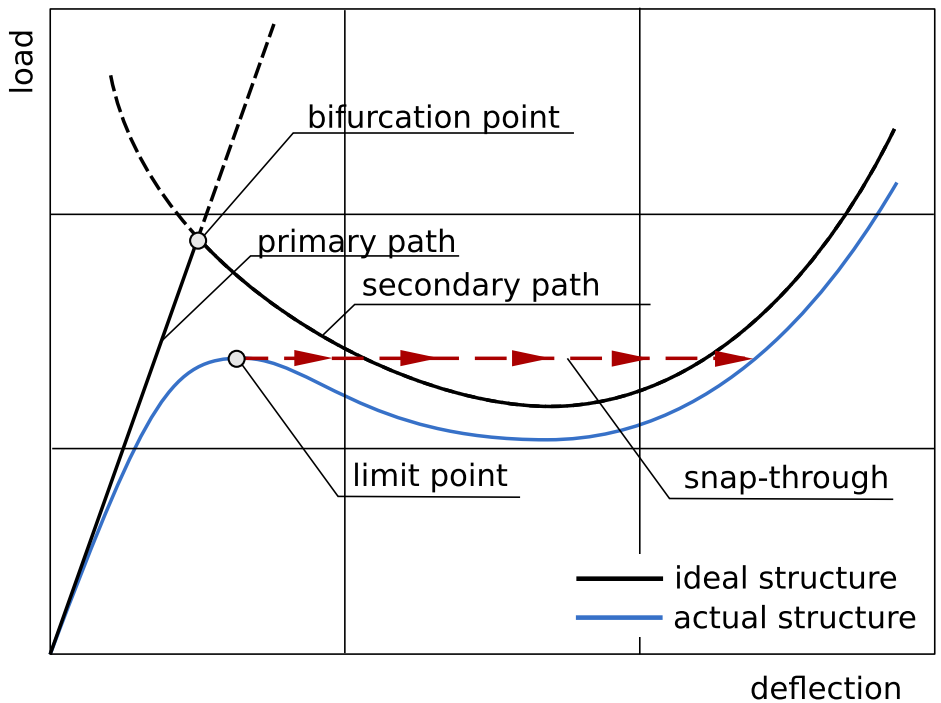
## Cylindrical tank under external pressure

- What about a theoretical solution?
- a theoretical solution gives us two intersecting curves



# Equilibrium path

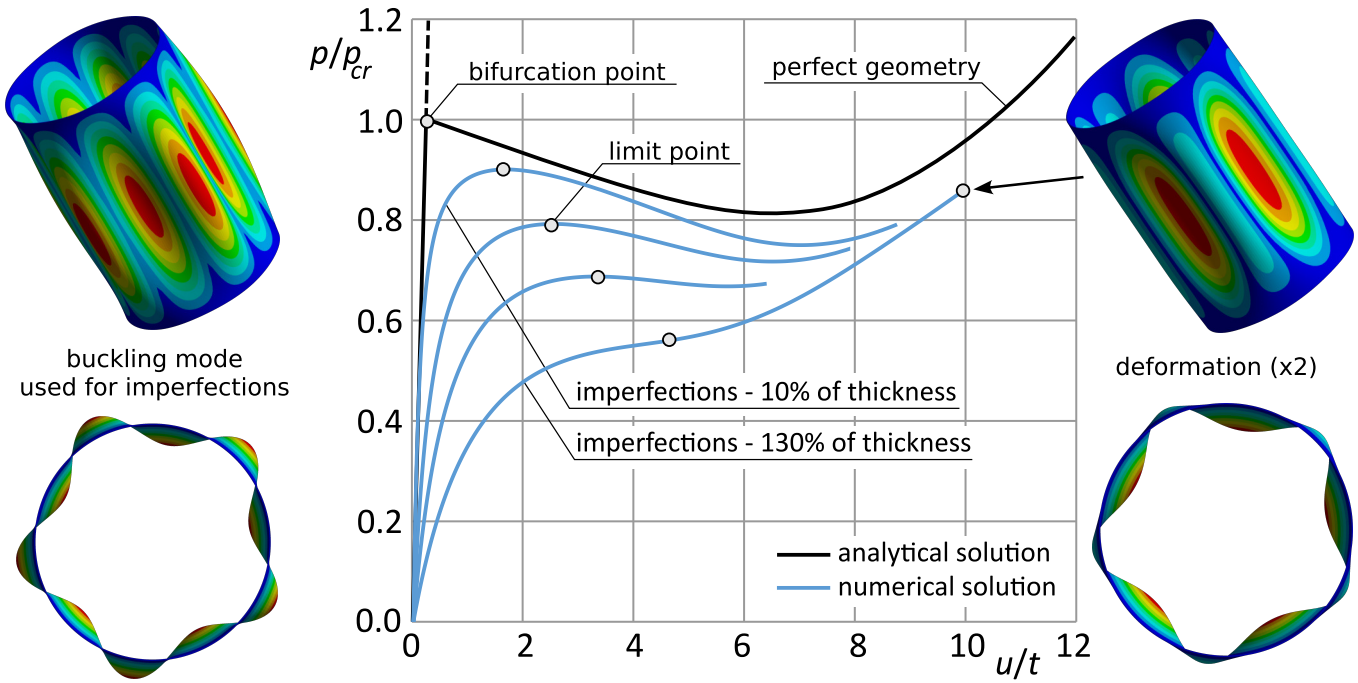
- the behaviour of the shell can be described in three ways: analytical, numerical and experimental
- the differences between these approaches are visible in load-deflection curves
- all three approaches complement one another



# Equilibrium path

## Influence of geometrical imperfections

Why is the limit point lower than the bifurcation point?



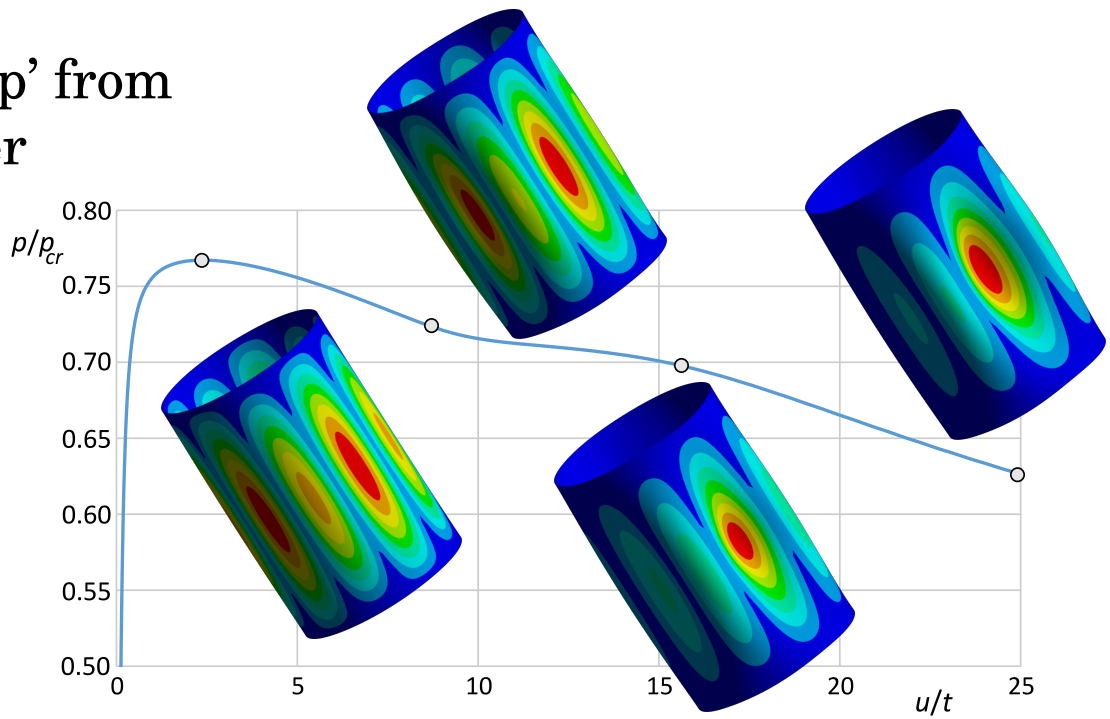

- the bifurcation point corresponds to the perfect structure
- the limit point is related to the imperfect structure

# Equilibrium path

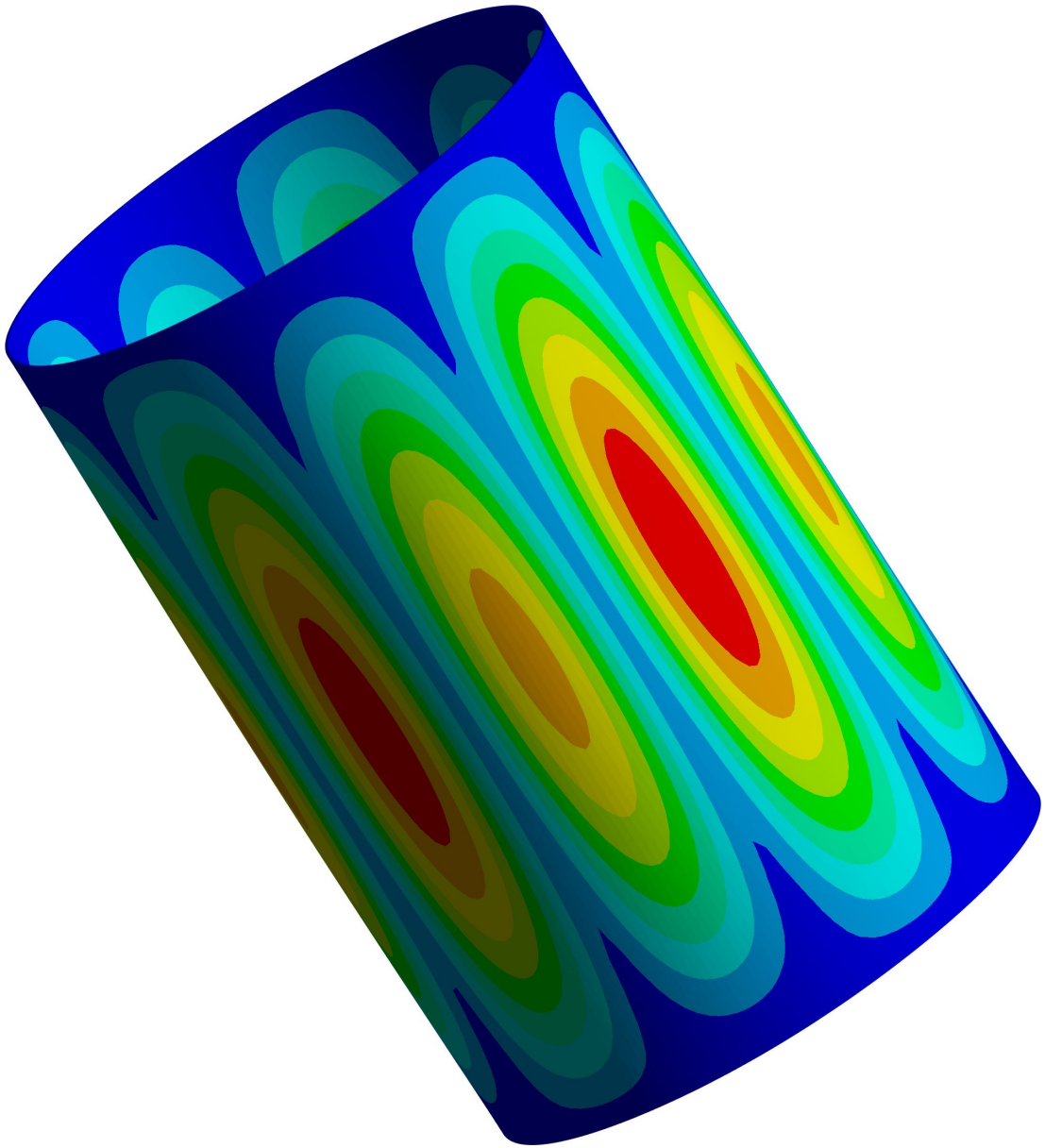
## Influence of geometrical imperfections

Is there only one solution?

- the behaviour of the shell depends not only on the magnitude of the imperfections, but also on their shape
- for different shapes, different equilibrium paths can be obtained
- the shell may 'jump' from one path to another



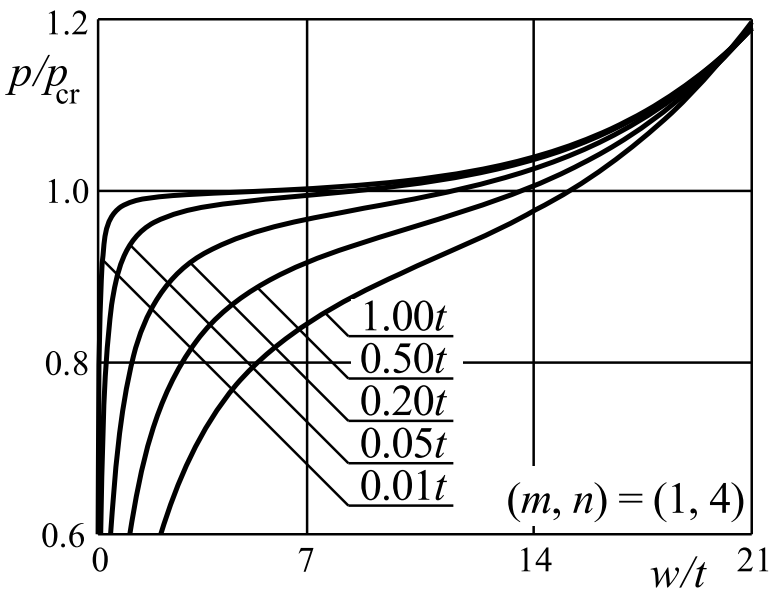
# Load carrying mechanism in shells

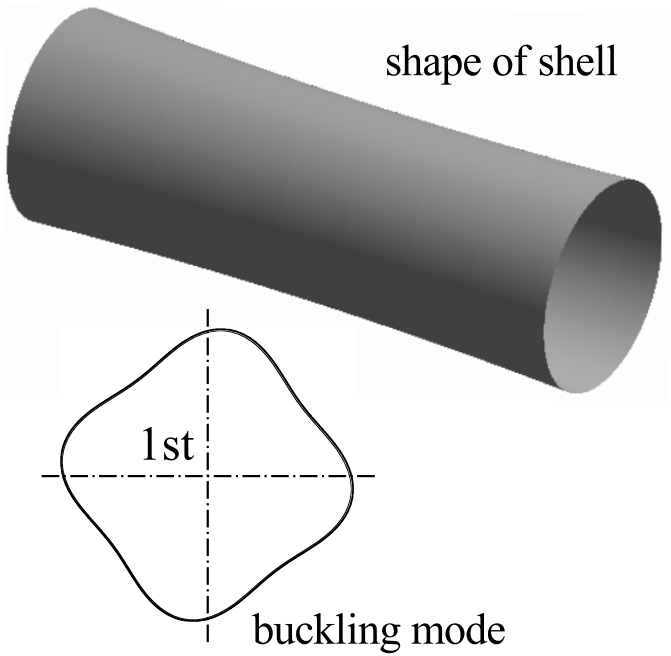


# Equilibrium path

## Influence of the magnitude of imperfections

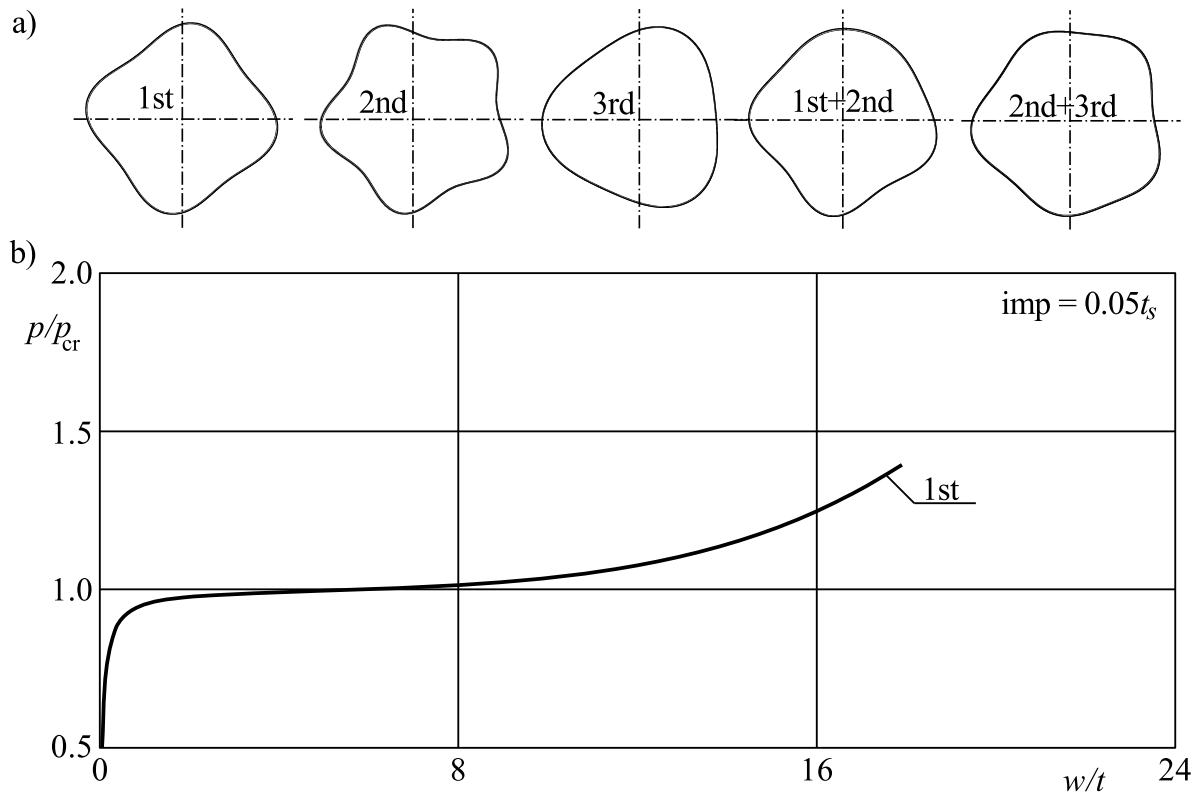
Let's consider a 'nearly' cylindrical shell under external pressure



# Equilibrium path

## Influence of the shape of imperfections



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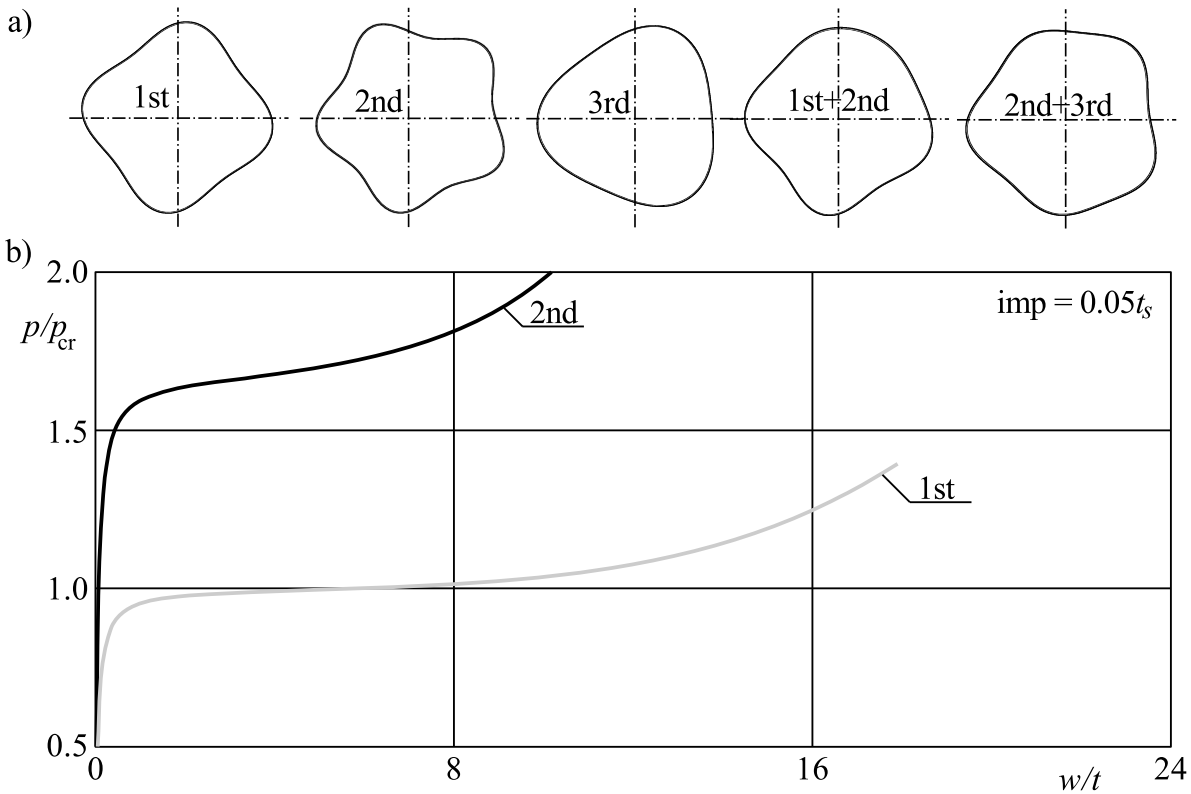

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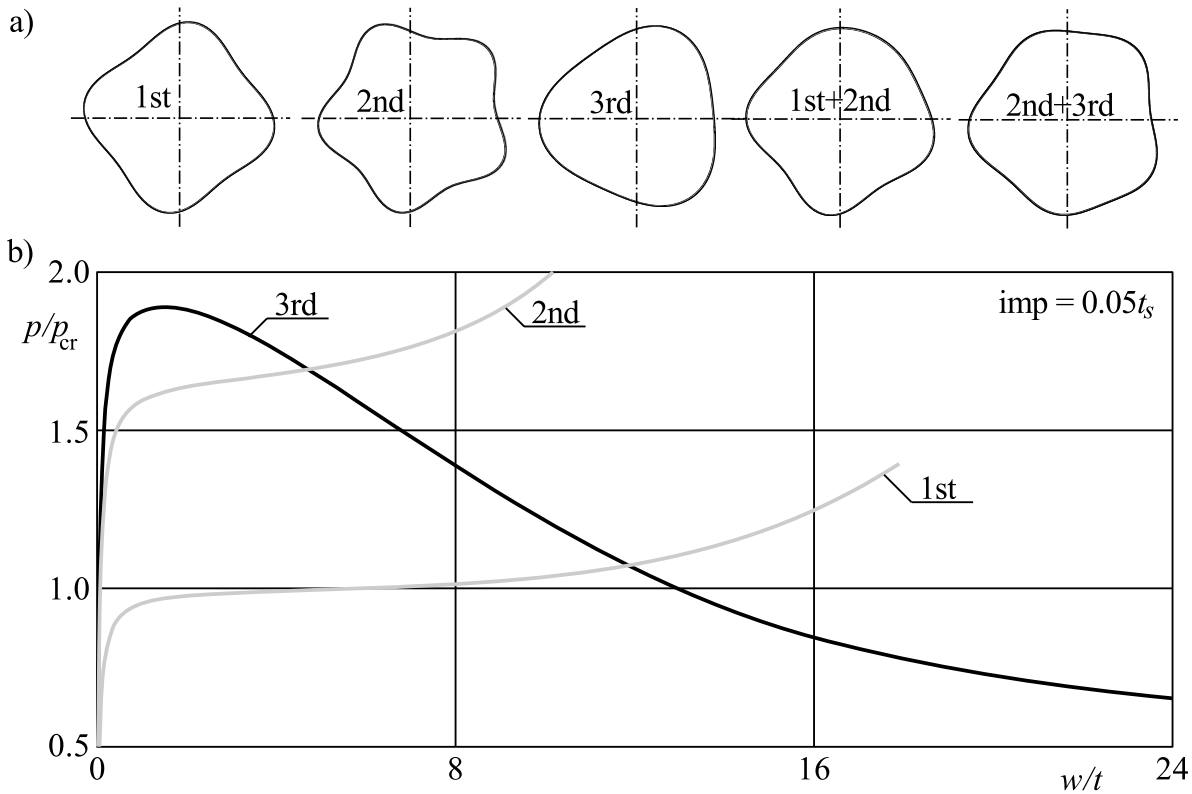

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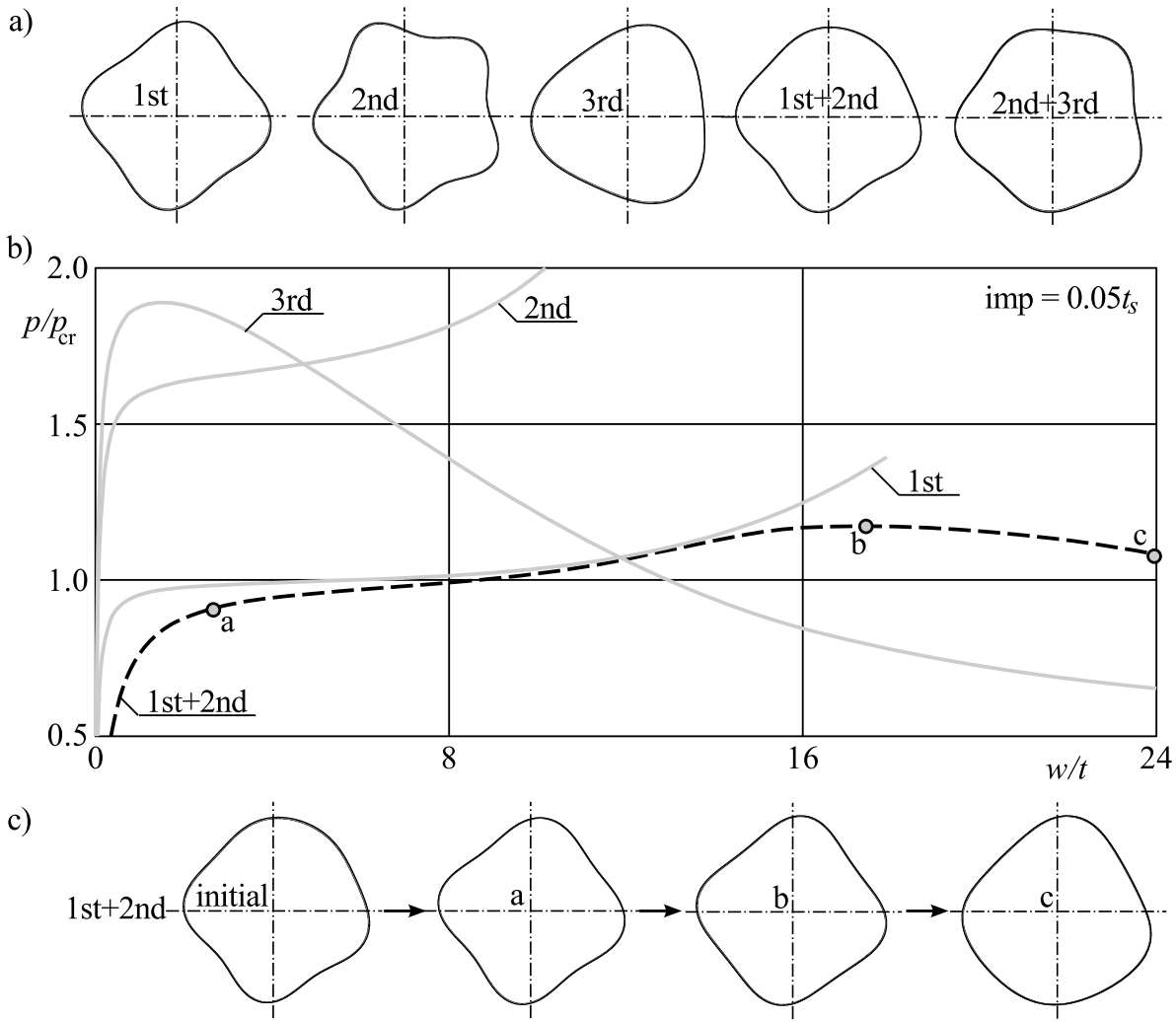

# Equilibrium path

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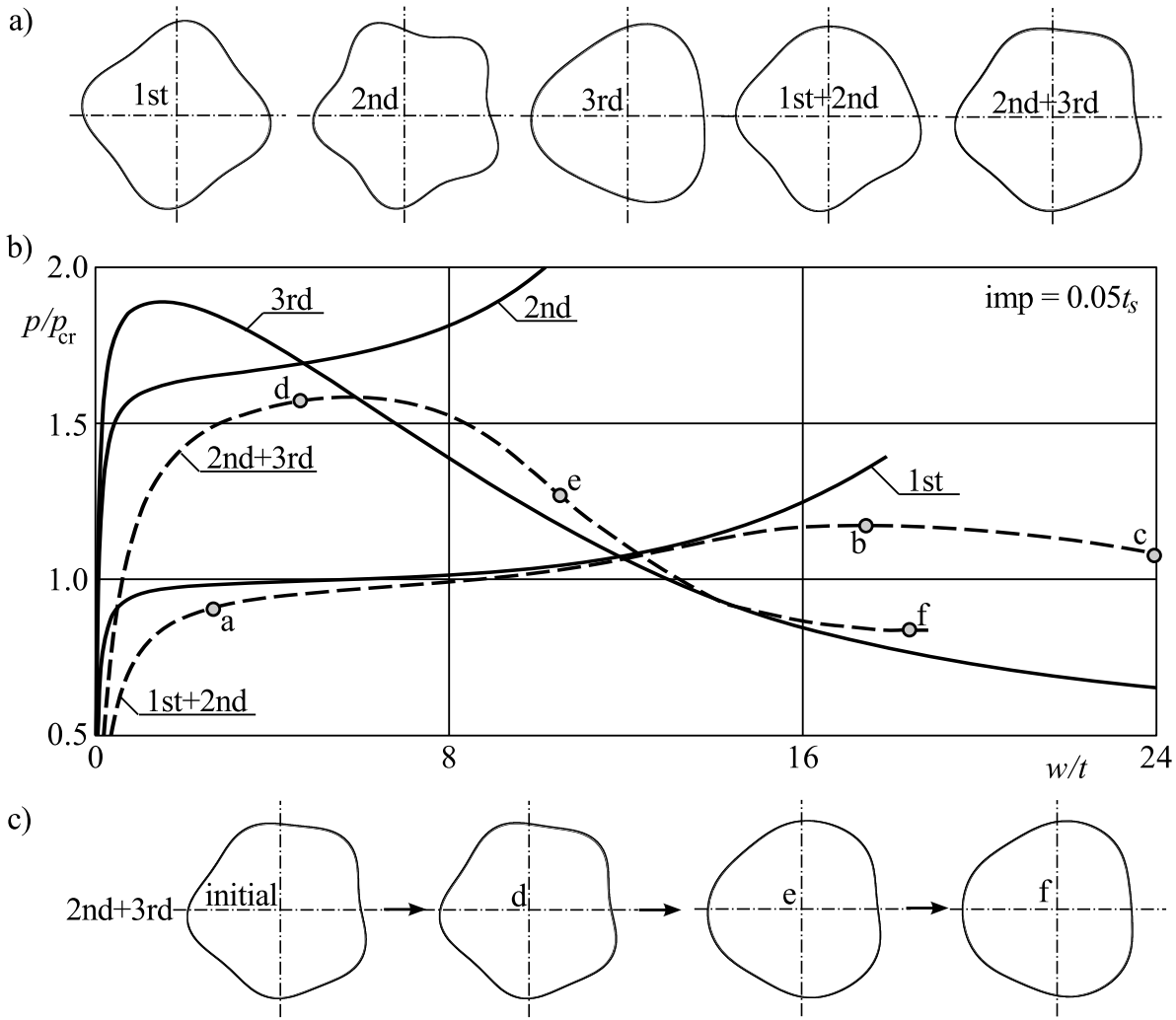

# Equilibrium path

## Practical example

- let's imagine a submersible descending to the seabed
- the greater the depth, the higher the pressure and deformation
- What load is safe, and under what assumptions?

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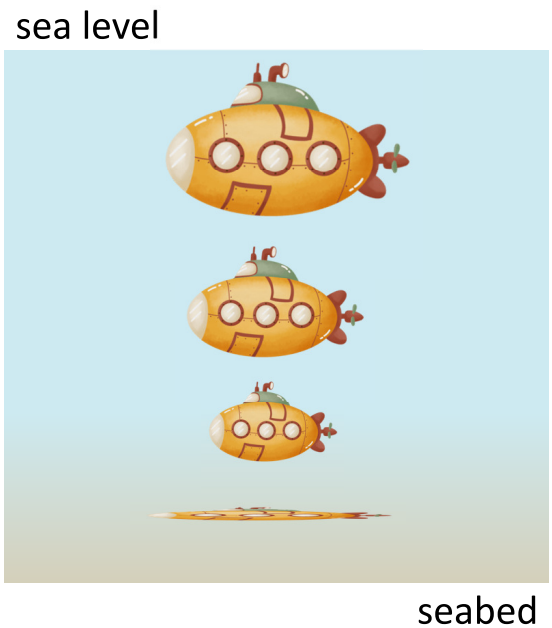
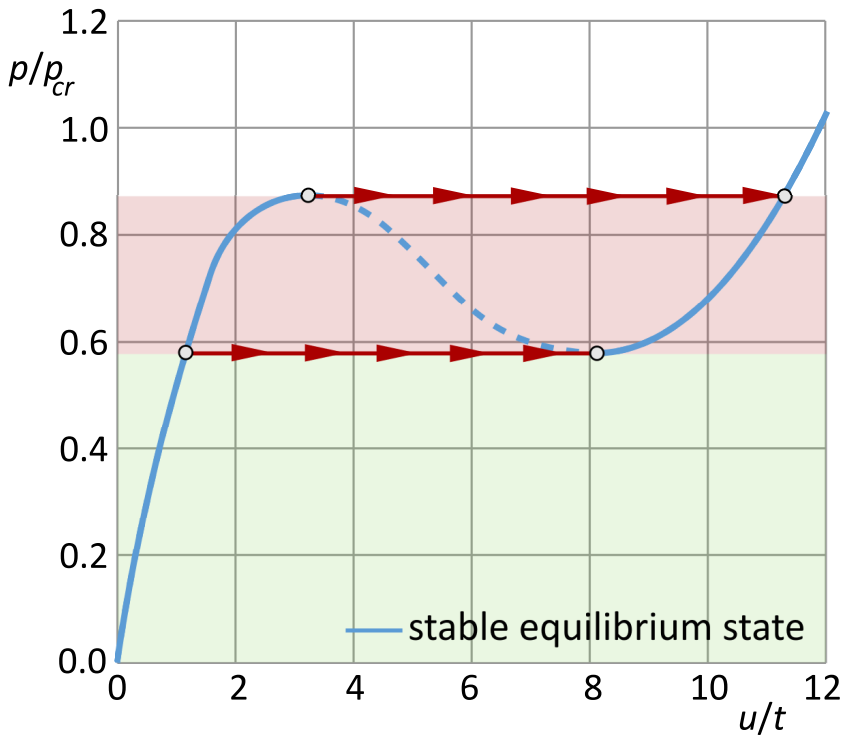
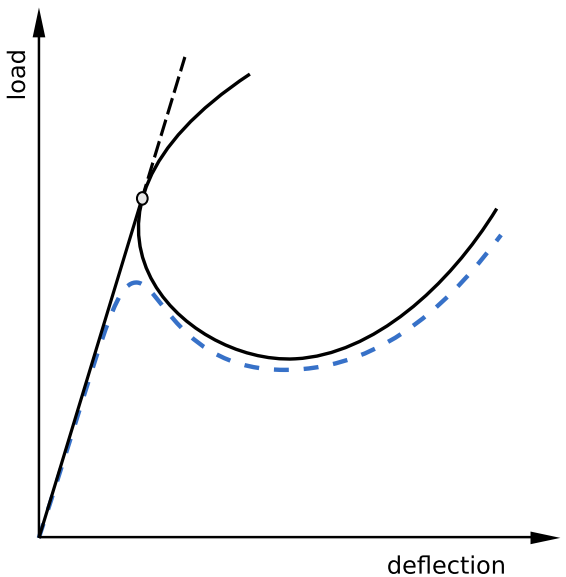


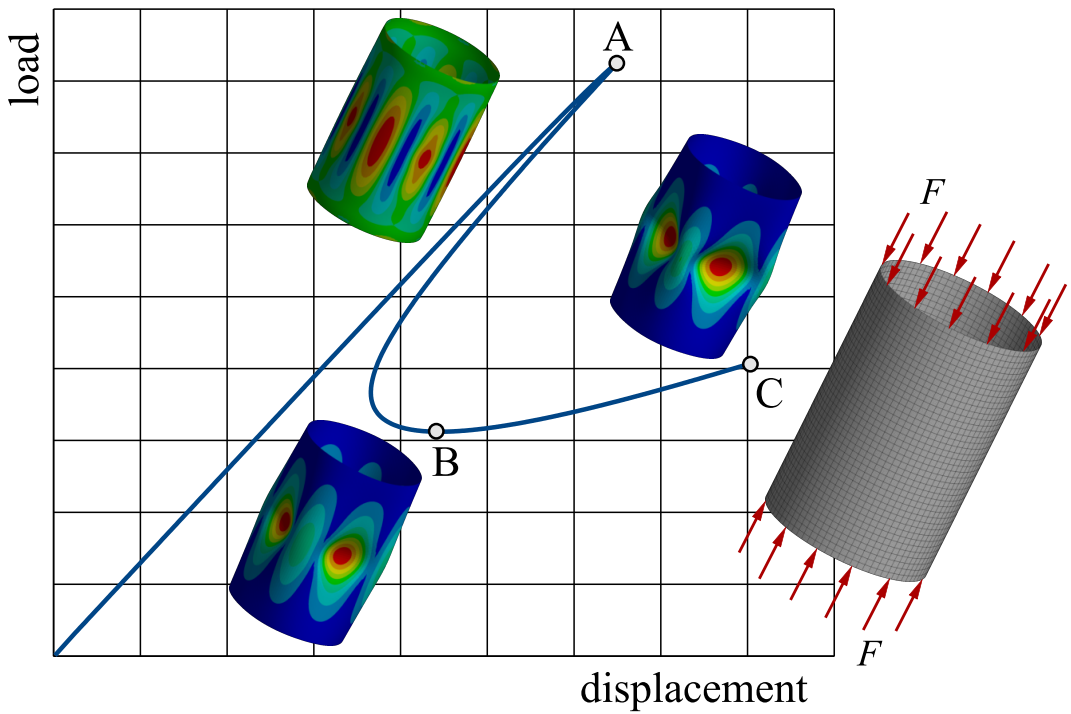



# Equilibrium path – other possible shapes

## Cylindrical shell under axial compression

- axially compressed cylindrical shells and spherical shells under external pressure have similar equilibrium paths
- in both cases the path is not stable; after the critical point is exceeded a sudden and substantial drop in load is observed; the structure collapses

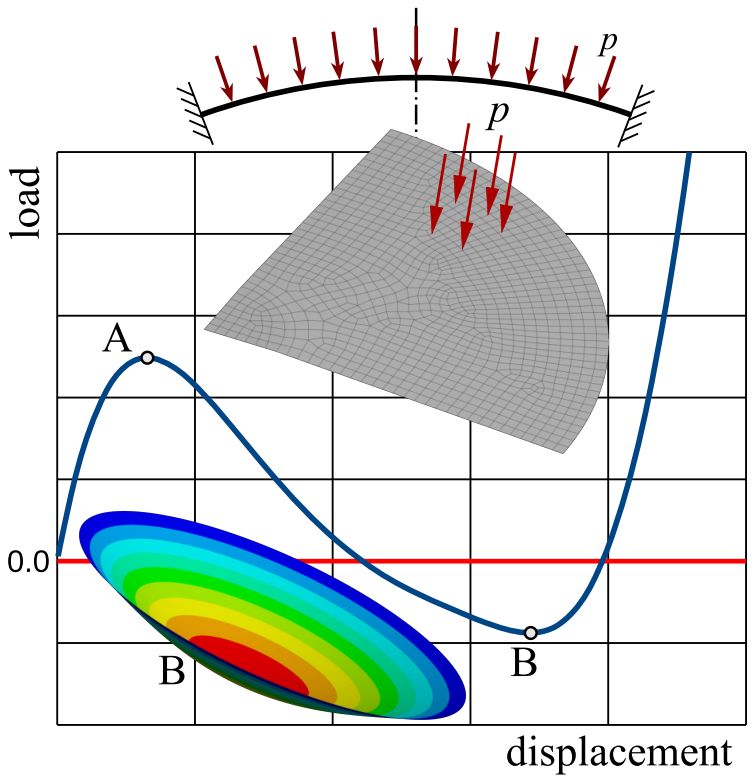


# Equilibrium path – other possible shapes

## Spherical cup under external pressure

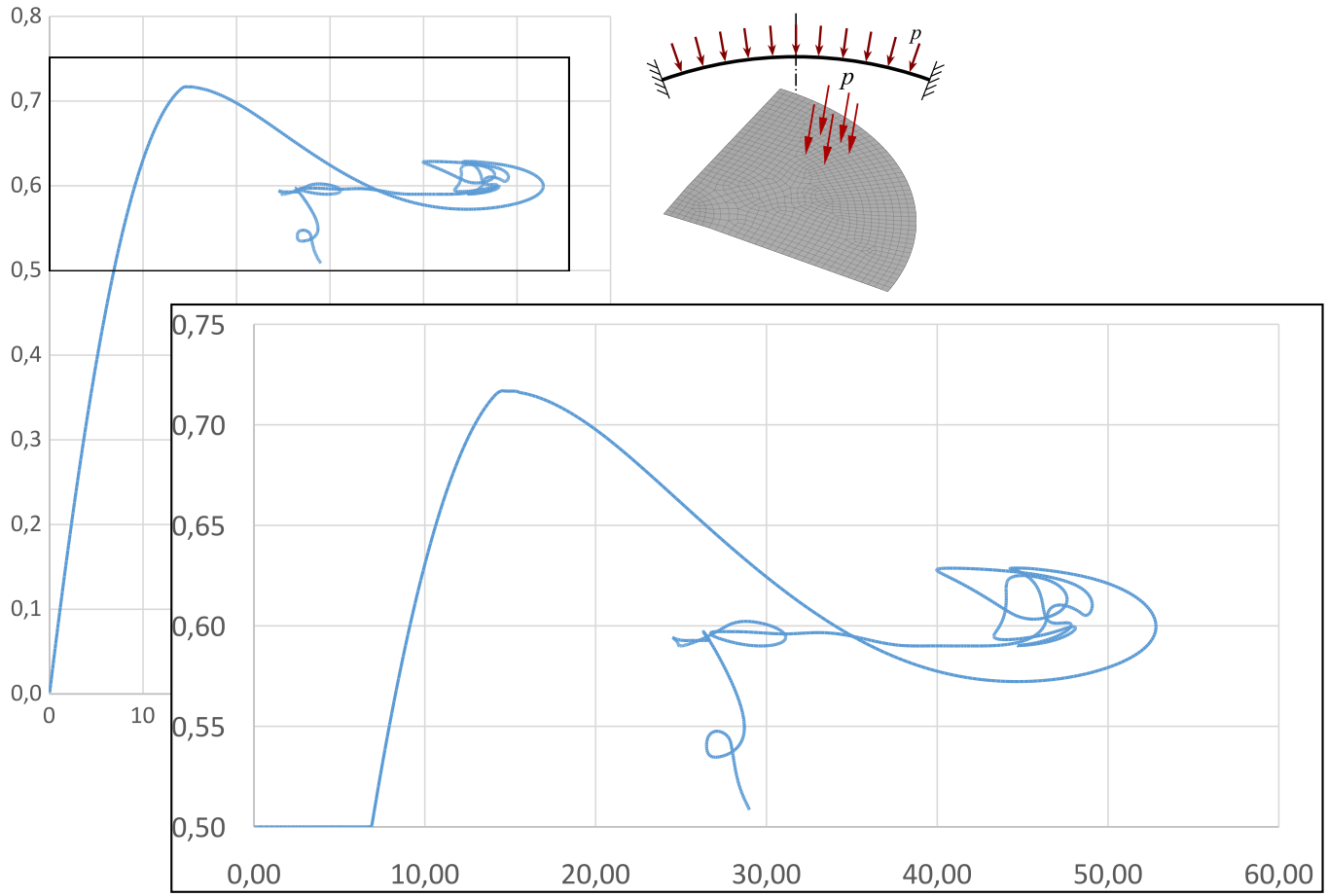
- a spherical cap under external pressure also has an unstable equilibrium path
- a snap-through is observed – to maintain the current deformation, a negative force must be applied



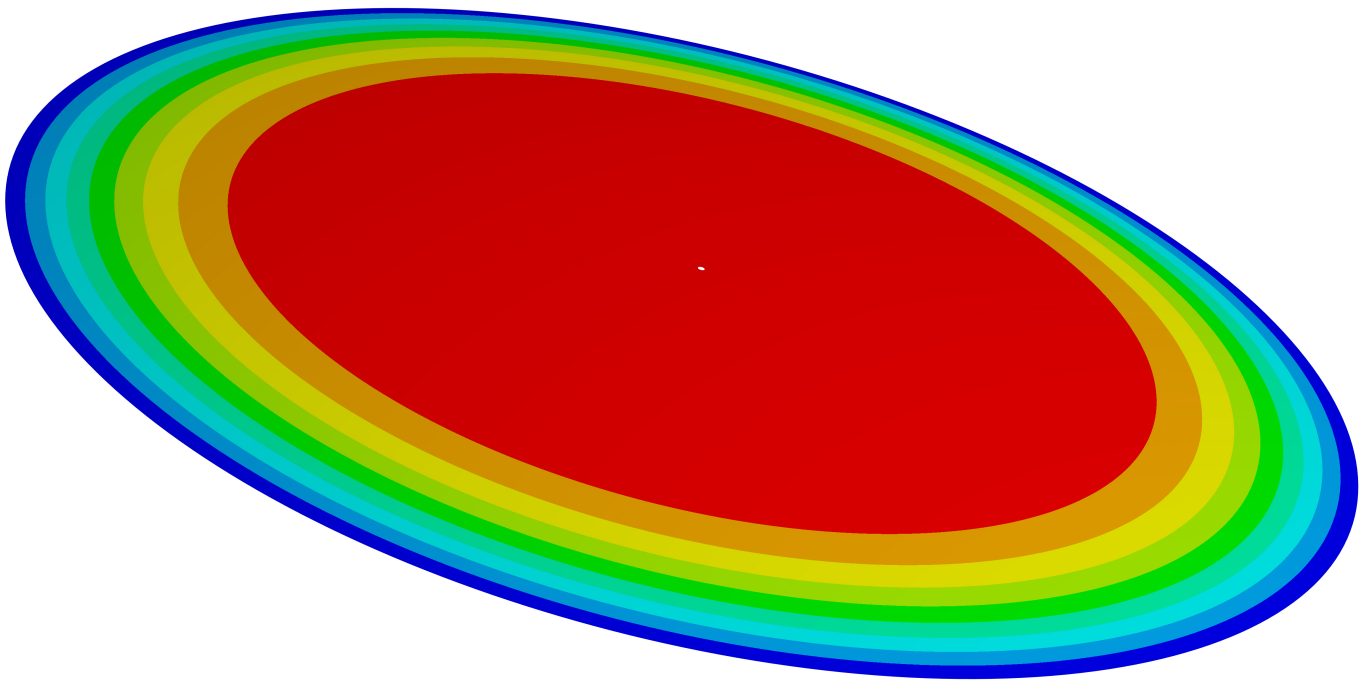



# Equilibrium path – other possible shapes

## Spherical cup under external pressure



# Equilibrium path – other possible shapes



# Equilibrium path – other possible shapes

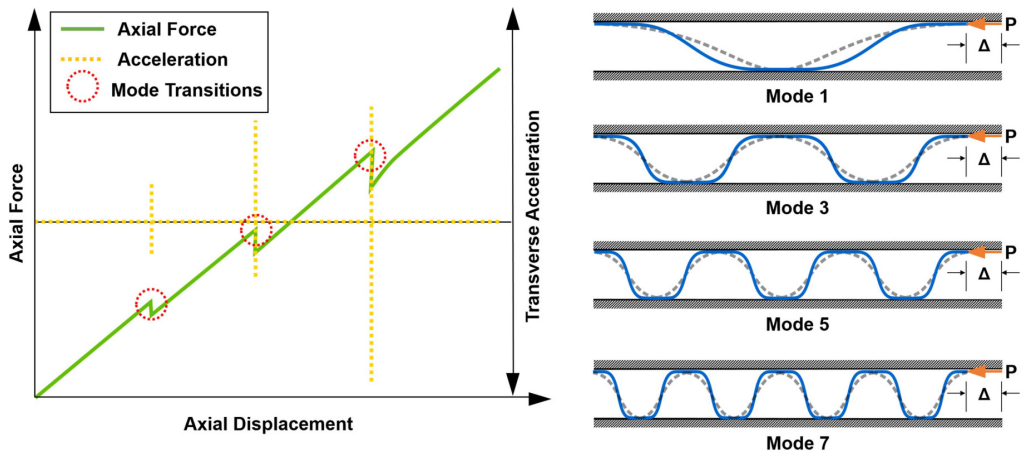


# Can buckling be a positive phenomenon?

Hu N., Liu S., Burgueño R. *Controlled elastic instabilities in cylindrical shells for energy harvesting devices*, SEMC 2016

- 'The sudden motion from elastic instabilities is a promising avenue to develop energy harvesting devices...{} Key to the concept is the ability to control the generation of snap-through transitions.'

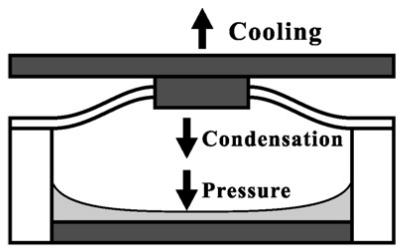
Liu S., Burgueño R. *Controlled elastic postbuckling of bilaterally constrained non-prismatic columns: application to enhanced quasi-static energy harvesters,* Smart Materials and Structures, 2016

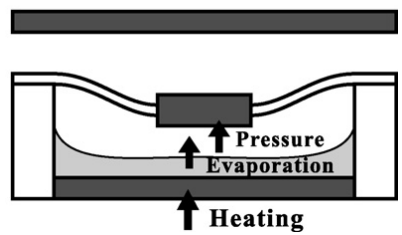


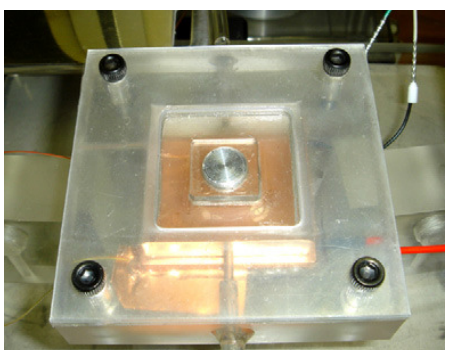
# Can buckling be a positive phenomenon?

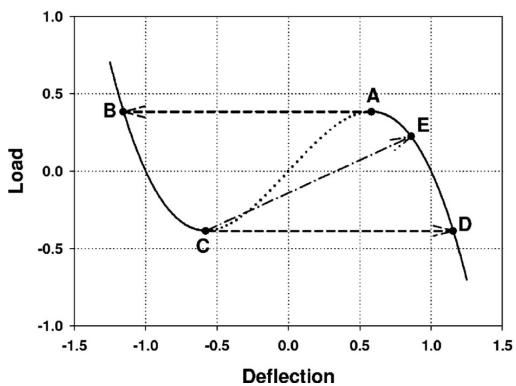
Song G.-E., Kim K.-H., Lee Y.-P. *Simulation and experiments for a phase-change actuator with bistable membrane*, Sensors and Actuators, 2007

- bistable membrane used as an actuator
- two phenomena are used:
  - heat transfer related to the gas–liquid phase change
  - snap-through buckling of shell



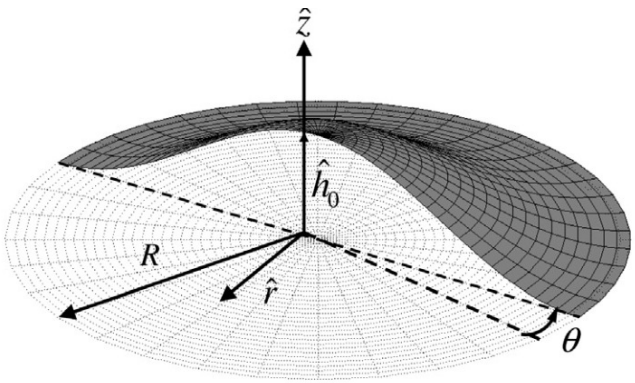



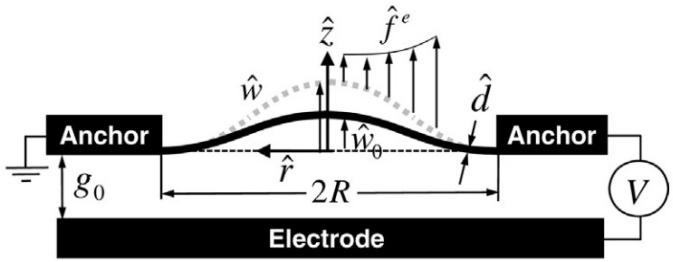


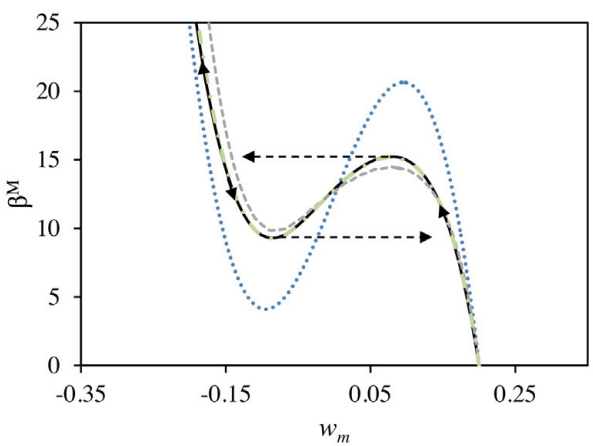
# Can buckling be a positive phenomenon?

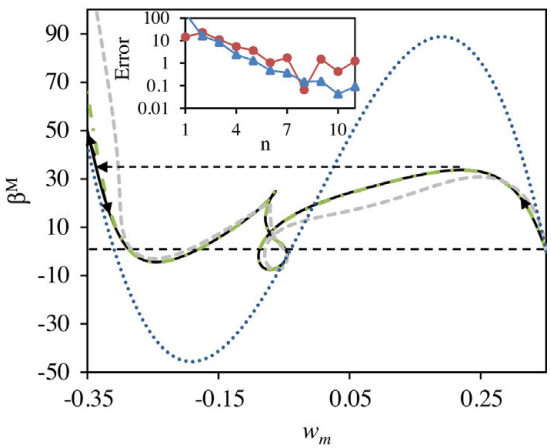
Song G.-E., Kim K.-H., Lee Y.-P. *Bistable behavior of electrostatically actuated initially curved micro plate*, Sensors and Actuators, 2016

- snap-through phenomenon is used to design an actuator
- two loads are considered: electrostatic and mechanical



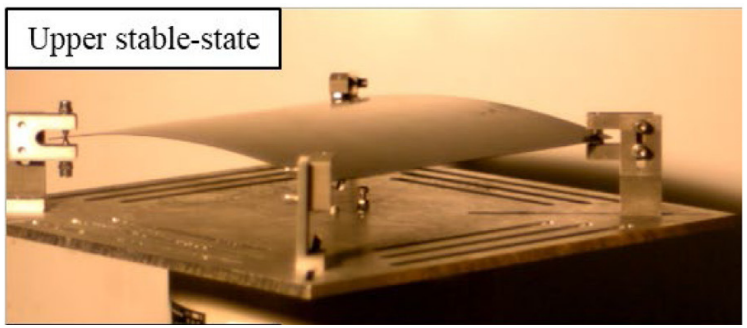



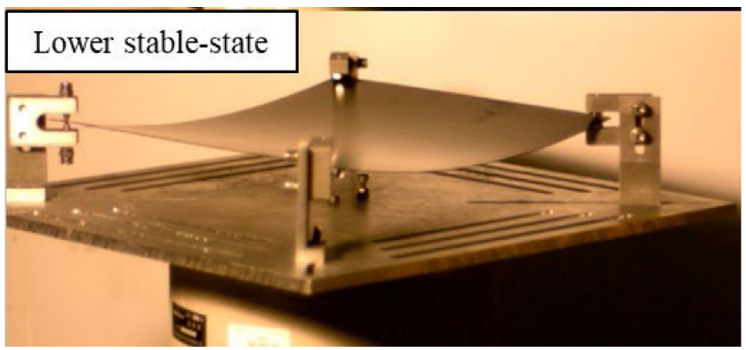



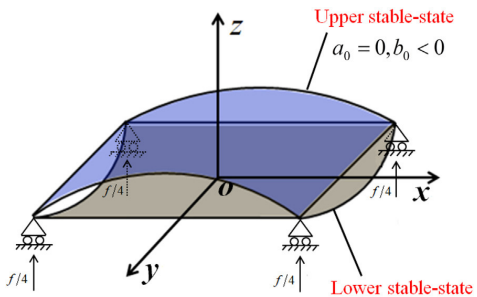
# Can buckling be a positive phenomenon?

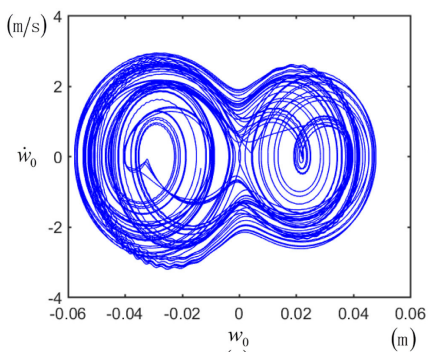
Guo X.T., Zhang W., Zhang Y.F. *Snap-through and nonlinear oscillations of bistable asymmetric FCSS composite laminated square plate subjected to excitation: Theoretical, numerical and experimental researches*, Thin-Walled Structures, 2022

- dynamic snap-through phenomenon is analysed in bistable asymmetric FCSS laminated square thin plate under the external excitation



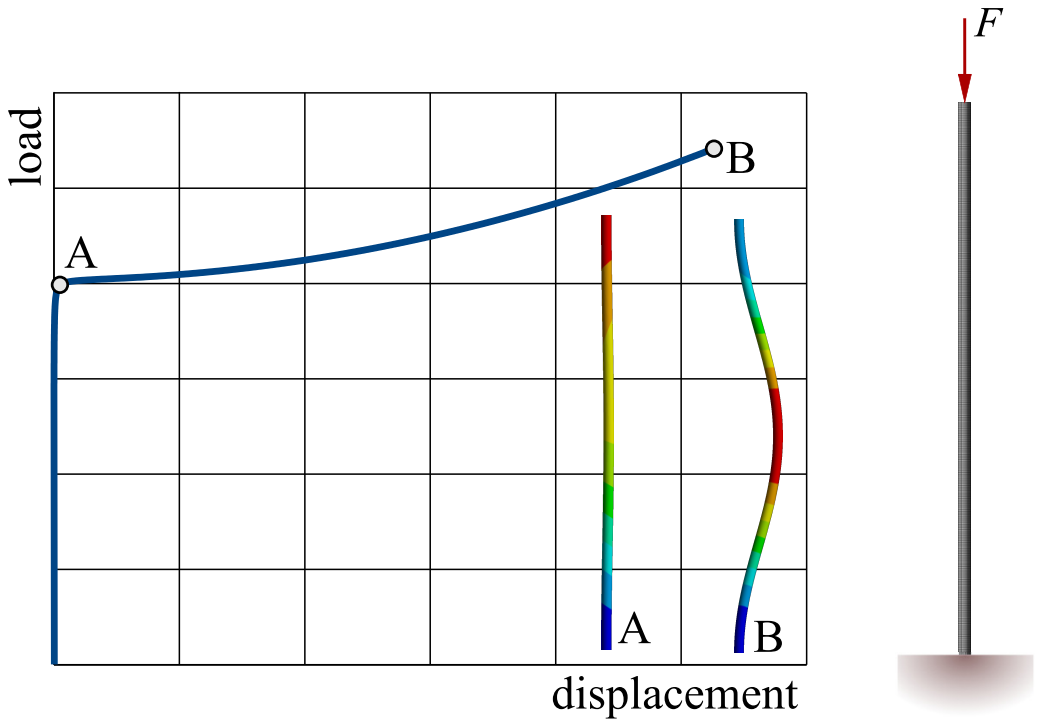



# Equilibrium path – other possible shapes

## Column under compression

- an axially compressed column has a stable equilibrium path
- this means that, once the critical point has been exceeded, further deformation is only possible if the load increases




Can a stable equilibrium path be achieved for a shell?