Energy methods in mechanics

Introduction

Each load acting on a structure which deforms performs the work W on this structure. The work is equivalent to the energy of elastic deformation U of the structure. Assuming that the load is static and the material follows the Hooke's law we can write that

$$U = W.$$

The energy of elastic deformation for simple load cases can be determined based on the following formulae

- tension (compression)

$$U_F = \frac{1}{2}\frac{F^2 l}{EA}; \quad \text{in general form} \quad U_F = \frac{1}{2}\int_0^l \frac{F^2 dx}{EA} \quad \text{or} \quad U_F = \int_0^l \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx \tag{1}$$

- torsion

$$U_T = \frac{1}{2}\frac{T^2 l}{GI_0}; \quad \text{in general form} \quad U_T = \frac{1}{2}\int_0^l \frac{T^2 dx}{GI_0} \quad \text{or} \quad U_T = \int_0^l \frac{GI_0}{2}\left(\frac{d\varphi}{dx}\right)^2 dx \tag{2}$$

- bending

$$U_M = \frac{1}{2}\frac{M^2 l}{EI}; \quad \text{in general form} \quad U_M = \frac{1}{2}\int_0^l \frac{M^2 dx}{EI} \quad \text{or} \quad U_M = \int_0^l \frac{EI}{2}\left(\frac{d^2 v}{dx^2}\right)^2 dx \tag{3}$$

Simple cases

Example 1
Determine the energy of elastic deformation stored in a simply supported beam loaded with transverse force F. The length of the beam equals l and the stiffness is defined with E and I.

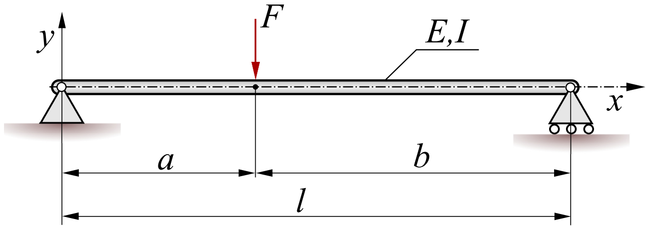

Fig. 1: Simply supported beam

For the beam shown above there are two internal loads: shear force V and bending moment M. Both loads do the work on a beam, however the work done by T is much smaller that this done by M. For this reason we can neglect the part of work done by the shear force and determine the energy based on the bending moment only.

Thus, the energy is expressed by the formula

$$U_M = \frac{1}{2}\int_0^l \frac{M^2(x)dx}{EI} \tag{4}$$

Since the material of the beam as well as its cross-section does not change through the whole length we can say that $EI = const.$ and put it in front of the integral

$$U_M = \frac{1}{2EI}\int_0^l M^2(x)dx \tag{5}$$

In the above formula we assumed that the bending moment is not constant since in most cases it is a function of the coordinate x, that is $M = M(x)$. From mechanics of materials we know that in this example the bending moment will be described with two different functions – one in the range from 0 to a, moving from the left hand side to the right, and in the range from 0 to b, moving from the right hand side to the left.

Before we start defining the functions of moments, we have to determine the values of reactions. Of course to do this the equations of equilibrium should be used. The convenient way is to determine the reactions from equations of moments according to the points of supports and then verify the correctness with the third equation that is the equation of forces according to axis y. Consequently we will obtain $R_A = (Fb)/l$ and $R_B = (Fa)/l$. Now we can start with defining the bending moments in the beam.

In the first portion of the beam $0 \leq x_1 \leq a$, according to the Fig. 2a. Writing the

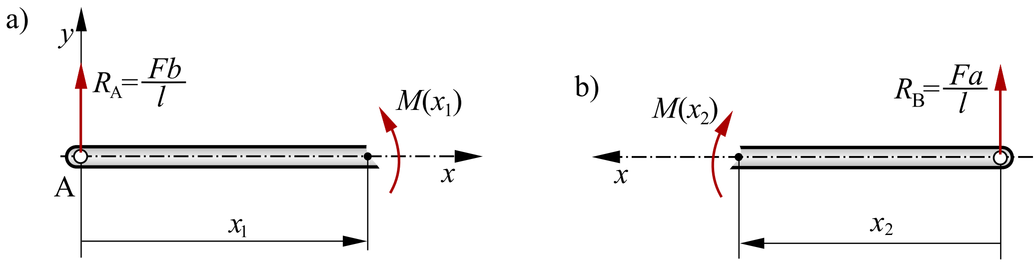

Fig. 2: First range

equation of equilibrium we can determine that the bending moment in the first portion is $M(x_1) = (Fbx_1)/l$. Similar, from Fig. 2b we can write, for the portion $b \geq x_2 \geq 0$, that $M(x_2) = (Fax_2)/l$. Knowing that the total energy of elastic deformation of the structural element is the sum of energies stored in its parts, we can write the equation (5) as

$$U_M = \frac{1}{2EI}\left[\int_0^a M^2(x_1)dx + \int_0^b M^2(x_2)dx\right] \tag{6}$$

After substituting the $M(x_1)$ and $M(x_2)$ from the above equation with the formulae determined before and after integration we will determine the energy of elastic deformation of the beam which is

$$U = \frac{F^2a^2b^2}{6EIl} \tag{7}$$

In the above example there is only one load which does the work on the beam, that is the force F. From the definition of work we know that it equals the force times the displacement of the point to which the force is applied. For the beam the displacement will be denoted as v, which is the deflection of the beam. We can write that the work W, and at the same time the energy, equals

$$W = U = \frac{1}{2}Fv \tag{8}$$

When we compare the two expression for energy we will determine the displacement of the point at which the force F is applied. That is

$$v = \frac{Fa^2b^2}{3EIl}. \tag{9}$$

This way it is possible to determine the displacement of the point of the beam at which the force is applied.

***

Example 2
Determine the energy of elastic deformation stored in a cantilever beam loaded at one end with the transverse force F and the bending moment M. The length of the beam equals l and the stiffness is defined with E and I.

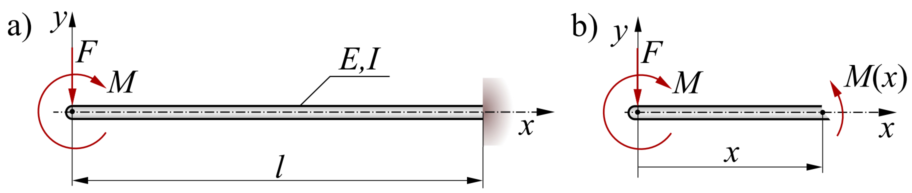

Fig. 3: Cantilever beam

Similar like in the previous example the stiffness of the beam is constant along the whole length. Thus we can writhe the formulae for the energy like

$$U_M = \frac{1}{2EI}\int_0^l M^2(x)dx \tag{10}$$

There will be only one function describing the bending moment since there is only one portion in which $0 \leq x \leq l$. According to Fig. 3b this function will have the form

$$M(x) = M - Fx \tag{11}$$

The moment has two components. Substituting this into the integral (10) we have

$$U_M = \frac{1}{2EI}\int_0^l (M - Fx)^2dx = \frac{1}{2EI}\left[\int_0^l (M)^2dx - 2\int_0^l (M \cdot Fx)dx + \int_0^l (Fx)^2dx\right] \tag{12}$$

Remembering that F and M are constant in this example, after integration the formula for the total potential energy of elastic deformation will be obtained in the form

$$U = \frac{M^2l}{2EI} - \frac{MFl^2}{2EI} + \frac{F^2l^3}{6EI} \tag{13}$$

Let's try to determine the displacement of the point of application of load similar like in the previous example. This time there are two loads which do the work on the beam that is the force F and the moment M. Than, according to the simple formula for determining the work of load we will get

$$\frac{1}{2}Fv + \frac{1}{2}M\theta = \frac{M^2 l}{2EI} - \frac{MFl^2}{2EI} + \frac{F^2 l^3}{6EI} \tag{14}$$

There is only one equation and two variables v and θ. Thus, in this case it is not possible to determine the displacement in the same way as in example 1. However there is a simple way to fix this by application of Castigliano theorem.

Application of Castigliano theorem

As it was shown on the lecture the Castigliano theorem gives us possibility to determine the displacements of selected points of the structure based on the elastic strain energy stored in it. The general formula has the form

$$\delta_i = \frac{\partial U}{\partial P_i}$$

> *The partial derivative of the strain energy of a structure with respect to any load is equal to the displacement corresponding to that load.*

We have to remember that one can determine the *generalised displacement* δ_i of the point at which the *generalised load* P_i is applied. In practise it means that if we want to determine the displacement – deflection – of the beam we have to have the force applied to this point. Similar, if we want to determine the angle of rotation of the beam at some point we have to have the moment applied to this point.

Let's return to the Example 2 solved before. We were able to determine the energy of elastic deformation but it was not possible to determine the displacement and angle of rotation of the end of the beam. With Castigliano theorem it is easy. It is enough to calculate partial derivatives of the energy with respect to a proper variable. The energy according to (13) equals

$$U = \frac{M^2 l}{2EI} - \frac{MFl^2}{2EI} + \frac{F^2 l^3}{6EI}$$

To determine the displacement v of the end of the beam we have to calculate the derivative with respect to the force F

$$v = \frac{\partial U}{\partial F} = \frac{\partial}{\partial F}\left[\frac{M^2 l}{2EI} - \frac{MFl^2}{2EI} + \frac{F^2 l^3}{6EI}\right] = \frac{Fl^3}{3EI} - \frac{Ml^2}{2EI} \tag{15}$$

Similarly, to determine the angle of rotation θ we have to calculate the derivative with respect to the moment M

$$\theta = \frac{\partial U}{\partial M} = \frac{\partial}{\partial M}\left[\frac{M^2 l}{2EI} - \frac{MFl^2}{2EI} + \frac{F^2 l^3}{6EI}\right] = \frac{Ml}{EI} - \frac{Fl^2}{2EI} \tag{16}$$

***

The Castigliano theorem allows us to find the displacement of a point at which the load is applied and the direction of displacement corresponds to the direction of the action of load. To find the displacement of other points or the displacements in direction other than the action of load a *fictitious* or *dummy* load has to be applied to the structure the value of which equals 0. The problem is solved in a usual way and the result is a function of an actual and fictitious load. By setting the fictitious load equal to 0 the displacement under the actual load is obtained.

Example 3
As an example lets find the displacement components of point 'c' of two bars shown in Fig. 4a and loaded with horizontal force F. Both bars have the same cross-section A and are made of the same material described by the Young modulus E.

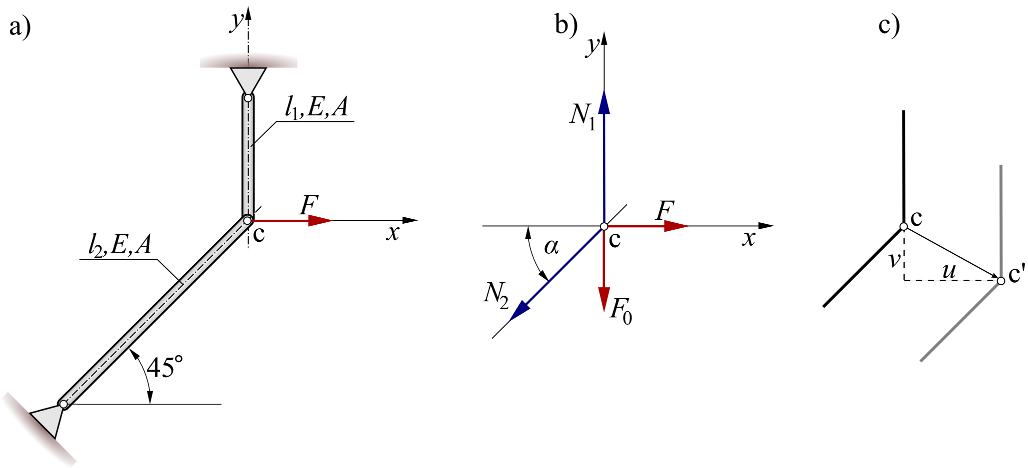

Fig. 4: Two bars

Since there is only a horizontal force F the horizontal displacement u can be determined only. To determine the vertical displacement v a fictitious vertical load F_0 has to be added. The diagram of all forces acting in the system is shown in Fig. 4b.

The energy of elastic deformation of the system of bars will be the sum of energies stored in both bars $U = U_1 + U_2$. Since both bars are in pure tension (compression) state the particular energies will have the form

$$U_1 = \frac{N_1^2 l_1}{2EA} \quad \text{and} \quad U_2 = \frac{N_2^2 l_2}{2EA} \tag{17}$$

We have to start with determining the internal forces N_1 and N_2 using the equations of equilibrium. Assuming that the angle α equals 45° we can write

$$\begin{aligned} \sum F_x = 0 \;\rightarrow\; N_2 \cos\alpha + F = 0 \;\rightarrow\; N_2 = \sqrt{2}F \\ \sum F_y = 0 \;\rightarrow\; N_1 - N_2 \sin\alpha - F_0 = 0 \;\rightarrow\; N_1 = F + F_0 \end{aligned} \tag{18}$$

Thus the elastic strain energy of the system is

$$U = U_1 + U_2 = \frac{(F+F_0)^2 l_1}{2EA} + \frac{(\sqrt{2}F)^2 l_2}{2EA} = \frac{F^2 l_1}{2EA} + \frac{F F_0 l_1}{EA} + \frac{F_0^2 l_1}{2EA} + \frac{F^2 l_2}{EA} \tag{19}$$

To determine the components of displacements partial derivatives with respect to particular forces have to be calculated. Since the fictitious load equals 0 all components containing this force also equals 0. Than we have

$$\begin{aligned} u_c = \frac{\partial U}{\partial F} = \frac{F l_1}{EA} + \left(\frac{F_0 l_1}{EA}\right)^{=0} + \frac{2F l_2}{EA} = \frac{F l_1}{EA} + \frac{2F l_2}{EA} \\ v_c = \frac{\partial U}{\partial F_0} = \frac{F l_1}{EA} + \left(\frac{F_0 l_1}{EA}\right)^{=0} = \frac{F l_1}{EA} \end{aligned} \tag{20}$$

One can see that this approach is much easier than the geometrical approach presented on the first course of mechanics of materials.

***

The function of bending moment used to defined the energy of elastic deformation may contain number of components depending on the number of loads applied to the structure. In example 2 there were two components (see (11)). This function is then square up which may generate a large number of components describing the energy.

To simplify the integration procedure the differentiation can be made before. Following the rules of calculus we can write

$$\delta_i = \frac{\partial}{\partial P_i}\int \frac{M(x)^2}{2EI}dx = \int \frac{M(x)}{EI}\frac{\partial M(x)}{\partial P_i}dx \tag{21}$$

The above formula is called *modified Castigliano's theorem.* Since there are usually two kind of generalised displacements to determine, that is v and θ, we can write

$$v = \frac{\partial U}{\partial F} = \int \frac{M(x)}{EI}\frac{\partial M(x)}{\partial F}dx \qquad \text{and} \qquad \theta = \frac{\partial U}{\partial M} = \int \frac{M(x)}{EI}\frac{\partial M(x)}{\partial M}dx \tag{22}$$

Advantages of this approach will be presented now on the modified version of Example 2 to which and additional force is added in the mid-length of the beam.

Example 4

Determine the displacement of the free end of a cantilever beam loaded with transverse forces F_1 and F_2, and the bending moment M. The length of the beam equals l and the stiffness is defined with E and I.

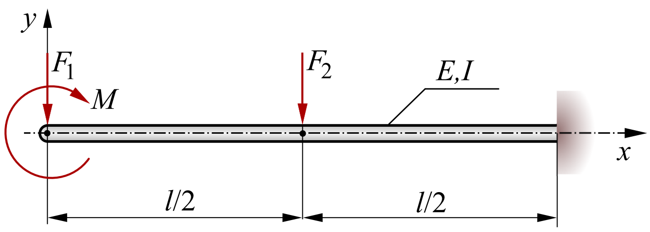

Fig. 5: Cantilever beam

There are two portions into which the beam can be divided. The first portion described by the moment $M(x)^I$ in which $0 \le x \le l/2$ and the second portion described by the moment $M(x)^{II}$ in which $l/2 \le x \le l$. From equations of equilibrium written for each portion there is

$$M(x)^I = M - F_1 x \qquad \text{and} \qquad M(x)^{II} = M - F_1 x - F_2\left(x - \frac{l}{2}\right) \tag{23}$$

The energy of elastic deformation will be the sum of energies stored in both parts of the beam $U = U^I + U^{II}$. With the classical approach both functions of moments must be square up and consequently we obtain 3 components from the first and 10 component from the second formula. This number will be diminished if the modified Castigliano theorem is applied.

In our case the vertical displacement of the free end of the beam is to be determined. At this point the force F_1 works the direction of which corresponds with the displacement which is looked for. For this reason the derivative of the energy must be calculated with respect to force F_1. We have thus

$$v = \frac{\partial U}{\partial F_1} = \int_0^{l/2} \frac{M(x)^I}{EI}\frac{\partial M(x)^I}{\partial F_1}dx + \int_{l/2}^{l} \frac{M(x)^{II}}{EI}\frac{\partial M(x)^{II}}{\partial F_1}dx \tag{24}$$

In the above equation we will need the formulae for bending moments $M(x)^I$ and $M(x)^{II}$ defined above (see (23)) and theirs derivatives. Let's calculate the derivatives then:

$$\begin{aligned} M(x)^I &= M - F_1 x \;\rightarrow\; \frac{\partial M(x)^I}{\partial F_1} = -x \\ M(x)^{II} &= M - F_1 x - F_2\left(x - \frac{l}{2}\right) \;\rightarrow\; \frac{\partial M(x)^{II}}{\partial F_1} = -x \end{aligned} \tag{25}$$

Substituting the above to Eq. (24) we will obtain

$$v = \frac{1}{EI}\left[\int_0^{l/2}(M - F_1 x)(-x)dx + \int_{l/2}^{l}(M - F_1 x - F_2(x - l/2))(-x)dx\right] \tag{26}$$

Solution of this equation will give the value of the vertical displacement of the free end of the beam.

***

The energy approach presented above allows to solve easily statically indeterminate problems too. The only thing that has to be noticed is that the displacement at the support is equal to 0. Than, if it is necessary to determine the reaction at the support it is enough to solve the following equation

$$v_{R_i} = \frac{\partial U}{\partial R_i} = 0 \tag{27}$$

The above formula is called the *Castigliano-Menabrea theorem*. The condition that has to be fulfilled to use it is that the energy of elastic deformation must be expressed as a function of the unknown reaction R_i.

Example 5
Let's consider cantilever beam of the length l supported at free end (point B) with a roller support and loaded with the force intensity q (see Fig. 6a). There are 4 unknown reaction and 3 equations of equilibrium that can be written. The fourth equation will be derived from the energy method.

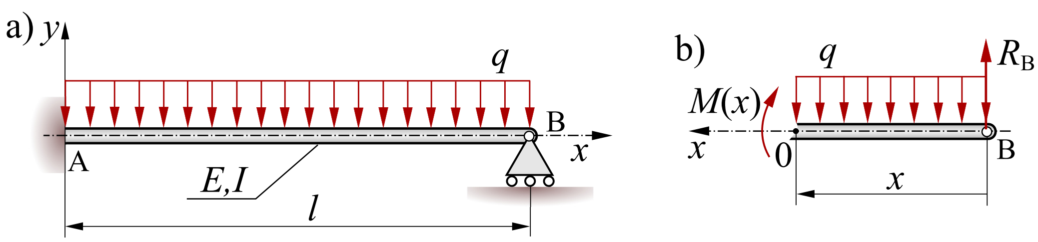

Fig. 6: Cantilever beam

To determine the moment equation let's cut the beam at the distance x from the right hand side end. From equation of equilibrium, according to Fig. 6b, we have

$$\sum M_0 = 0 \;\rightarrow\; M(x) + \frac{1}{2}qx^2 - R_B x = 0 \;\rightarrow\; M(x) = R_B x - \frac{1}{2}qx^2 \tag{28}$$

Partial derivative of the moment with respect of R_B is

$$\frac{\partial M(x)}{\partial R_B} = x$$

From the Castigliano-Menabrea theorem we have

$$\frac{\partial U}{\partial R_B} = \int \frac{M(x)}{EI} \frac{\partial M(x)}{\partial R_B} dx = 0$$

Substituting particular components and assuming that $EI = const.$ there is

$$\int_0^l \left(R_B x - \frac{1}{2} q x^2 \right) x dx = 0 \quad \rightarrow \quad R_B = \frac{3}{8} q l$$

Knowing the reaction R_B one can determine the remaining reactions.

References

1. Goodno GJ., Gere JM. *Mechanics of Materials*, Cengage Learning, Australia, 2018
2. Belyaev NM. *Strength of Materials (5th ed.)*, MIR Publishers, Moscow, 1979
3. Timoshenko S. *Strength of Materials, part II: Advanced Theory and Problems*, D. Van Nostrand Company, Inc., New York, 1947