Thick-walled cylinders
Lame equation

We will start the stress analysis in a thick-walled cylinder with formulating the assumptions. First of all the considerations will be conducted for an open cylinder the length of which is much bigger than the external diameter. The stress will be considered far from the point of support. Then the following is truth:

- deformation is symmetrical with respect to the axis of revolution and does not change along the length of the cylinder,
- due to symmetry the shear stresses are not present,
- the cross-section remains plain after the load is applied.

Thus, we can assume that the cylinder is in the plane state of strain.

Let's consider the cylinder of the internal radius a and the external radius b. From the cylinder, with two parallel planes, the ring of the thickness equal to unity is cut as shown in Fig. 1a. It is assumed that in general case the cylinder can be loaded with the uniform pressure, both internal p_a and external p_b.

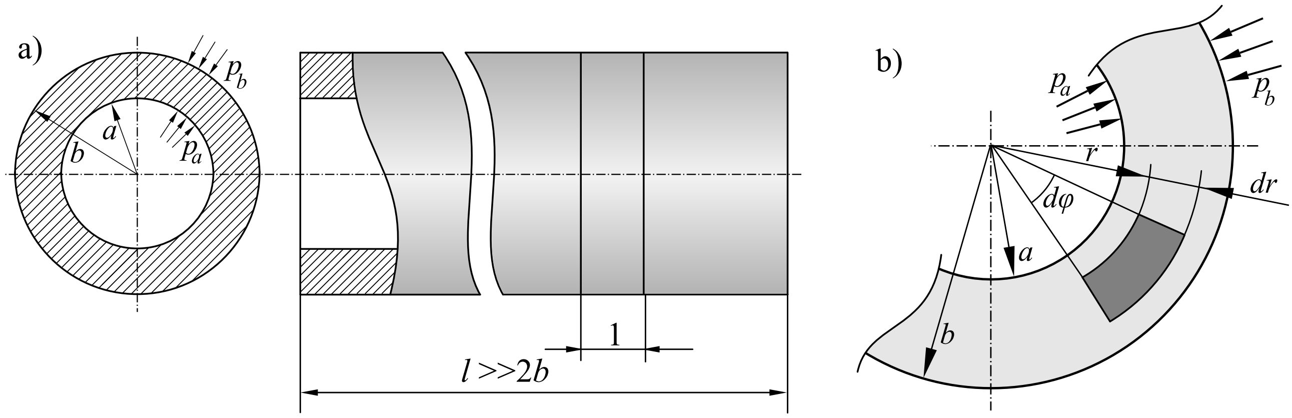

Fig. 1. Thick-walled cylinder: dimensions a); small element b)

From the ring, with two radial sections rotated relative to each other about the angle $d\varphi$, we cut a small element of the length dr (Fig. 1b). On each side of the element we can define the stress. Since the task is axisymmetrical there will be two components of stress: the radial stress σ_r and the hoop stress σ_t. Due to symmetry the hoop stress on both sides will be equal to each other. Instead, the radial stress changes since the coordinate r increases. On the inside face we will have $\sigma_r(r) = \sigma_r$, whereas on the outside one $\sigma_r(r + dr) \approx \sigma_r + d\sigma_r$.

To determine the value of the stress it is necessary to write the equation of equilibrium of a small element shown in Fig. 2a. It will be the equation of the sum of all forces projected on the bisector of the angle $d\varphi$. To obtain forces one has to multiply the stress by the area on which the stress acts.

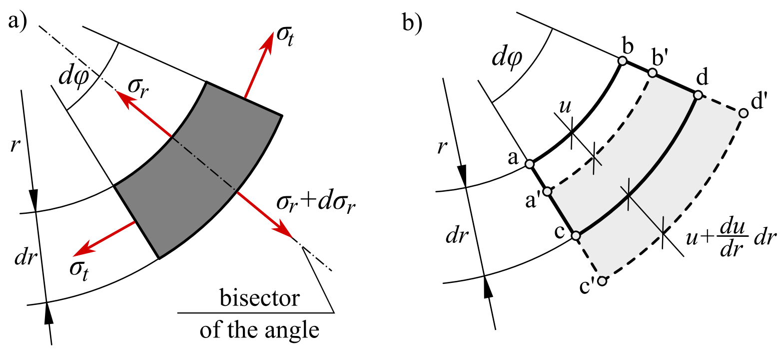

Fig. 2. Small element of the cylinder: stresses a); deformation b)

Since the thickness of the cylinder is equal to unity it is enough to multiply the stress by the length of a proper edge. Thus we will have

$$-\sigma_r r d\varphi + (\sigma_r + d\sigma_r)(r + dr)d\varphi - 2\sigma_t \sin\frac{d\varphi}{2}dr = 0 \tag{1}$$

It can be noted that the angle $d\varphi$ is very small and the sine of such an angle equals the angle. We can write then

$$\sin\frac{d\varphi}{2} \approx \frac{d\varphi}{2}$$

Increase of the stress $d\sigma_r$, being the differential of the function, can be written as

$$d\sigma_r = \frac{d\sigma_r}{dr}dr$$

Introducing the above to the equation (1) and after ordering we will get

$$-\sigma_r r d\varphi + \sigma_r r d\varphi + \sigma_r dr d\varphi + \frac{d\sigma_r}{dr} r dr d\varphi + \frac{d\sigma_r}{dr} dr^2 d\varphi - \sigma_t d\varphi dr = 0$$

After simplification, omitting the small of the second order (dr^2) and after dividing the equation by $d\varphi dr$ the result is

$$\sigma_r - \sigma_t + \frac{d\sigma_r}{dr} r = 0 \tag{2}$$

This is the only equation of equilibrium that can be written. However, since there are two variables to determine an additional equation is needed. It can be obtained by analysis of deformation of the element abcd of the cylinder (Fig. 2b). Under the load the cylinder will deform and the element abcd will translate and deform in the way that the arc ab will move about $u(r) = u$ and the arc cd about $u(r + dr) \approx u + du = u + (du/dr)dr$. Thus, the strains will appear in both directions, the radial one, and this will be denoted as ε_r, and the hoop one, denoted as ε_t. The first component of strains will cause the elongation of the elementary radius dr and the second component will cause the elongation of each selected arc in the element *abcd*. Let's try to write the expression for the radial and hoop strain using the definition of the normal strain.

Normal strain is defined as the ratio of the increase of the length and the initial length. In case of radial strain the initial length equals dr or ac. After deformation the segment will have the length *a'c'*. It can be noted that the increase of the length of the element ab (dr) will be equal to the change of the length of the elements *a'c'* and ac and this change will be equal to the difference between the displacement of point c and a. It can be written thus

$$\varepsilon_r = \frac{\Delta dr}{dr} = \frac{u + \frac{du}{dr}dr - u}{dr} = \frac{du}{dr}$$

To define the hoop strain let's analyse the length of the arc ab that before deformation equals $rd\varphi$. After deformation it changes into arc *a'b'* the length of which is $(r + u)d\varphi$. According to definition of normal strain we can write

$$\varepsilon_t = \frac{a'b' - ab}{ab} = \frac{(r + u)d\varphi - rd\varphi}{rd\varphi} = \frac{u}{r}$$

Thus, strains, the radial and the hoop one, in the thick-cylinder are defined as follows

$$\varepsilon_r = \frac{du}{dr} \quad \text{and} \quad \varepsilon_t = \frac{u}{r} \tag{3}$$

As it is seen both strains are functions of the displacement u. Thus, if we write the stress as a function of strains, both stress components will be also functions of one displacement. Then, by substituting them into the equation (2) the equation will be obtained in which only one unknown function will be present namely the displacement $u(r)$.

The expression combining the stress and strain is the Hooke's law. For two dimensional state of stress, with which we are dealing here, the law has the form

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_t\right)$$

$$\sigma_t = \frac{E}{1-\nu^2}\left(\varepsilon_t + \nu\varepsilon_r\right)$$

Substituting the expressions for strains into the above system we will get

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \\ \sigma_t &= \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) \end{aligned} \tag{4}$$

Now we can introduce the above relations into the equation (2). As can be seen after substitution each element of the equation will contain the constant value $E/(1-\nu^2)$. Since the left hand side of the equation is equate to zero the constant can be eliminated and not taken into account when substituting.

Before we start lets calculate the derivative of the radial stress. Having in mind that u is a function of the radius r, which is our variable, and using the quotient rule we have

$$\frac{d\sigma_r}{dr} = \frac{d}{dr}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) = \frac{d^2u}{dr^2} + \nu\left(\frac{\frac{du}{dr}r - u\frac{dr}{dr}}{r^2}\right) = \frac{d^2u}{dr^2} + \nu\frac{1}{r}\frac{du}{dr} - \nu\frac{u}{r^2}$$

Substituting the above as well as the stress expressed with the Hooke's law into the equation of equilibrium (2), after reordering and dividing by r we have

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{5}$$

This way the equation for determining the displacement u is obtained. The general solution of the above equation has the form

$$u = C_1 r + \frac{C_2}{r}$$

what can be proved if the first and second derivatives of it are calculated and put back into the equation (5). Substituting the expression for u into the Hooke's law, equation (4), and after ordering the solution of the problem is obtained that has the form of two equations allowing to determine the stress in the thick-walled cylinder.

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\frac{1-\nu}{r^2}\right] \\ \sigma_t &= \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\frac{1-\nu}{r^2}\right] \end{aligned} \tag{6}$$

It is worth to notice that the difference between the two equations is in the sign placed in front of the constant C_2. The above solution is a general one containing two constants C_1 i C_2. These constants can be determined depending on the boundary conditions. According to Fig. 3 there are three boundary conditions possible:

1. for $r = a$ stress $\sigma_r = p_a$ and for $r = b$ stress $\sigma_r = p_b$; it means that the cylinder is loaded by external pressure p_a and internal pressure p_b

2. for $r = a$ stress $\sigma_r = p_a$ and for $r = b$ stress $\sigma_r = 0$; it means that the cylinder is loaded only with the internal pressure p_a

3. for $r = a$ stress $\sigma_r = 0$ and for $r = b$ stress $\sigma_r = p_b$; it means that the cylinder is loaded only with the external pressure p_b

The constants determined from the boundary conditions are substituted into the equations (6) and this way the formulae for determining the stress distribution on the cross-section of the cylinder are determined. In all cases the only variable is r which vary in the range from a to b. The most general case is the first one. After determining the constants and substituting them into the equations we get

$$\begin{aligned} \sigma_r &= \frac{p_a a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) - \frac{p_b b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right) \\ \sigma_t &= \frac{p_a a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) - \frac{p_b b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right) \end{aligned} \tag{7}$$

Both equations differ in terms of signs in parentheses. Moreover, it is seen that by substituting $p_a = 0$ or $p_b = 0$, respectively, two other cases of solution can be obtained

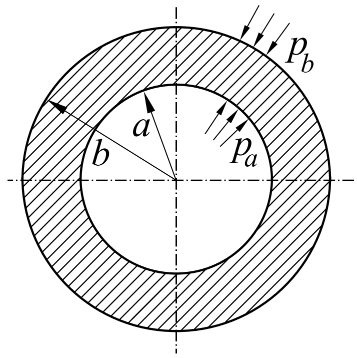

Fig. 3. Cross-section of the thick cylinder

Numerical examples

Example 1

Determine the stress distribution in a thick-walled hydraulic cylinder for which the maximum working pressure is $p_0 = 20$ MPa. Internal diameter $d = 40$ mm and external diameter $D = 50$ mm.

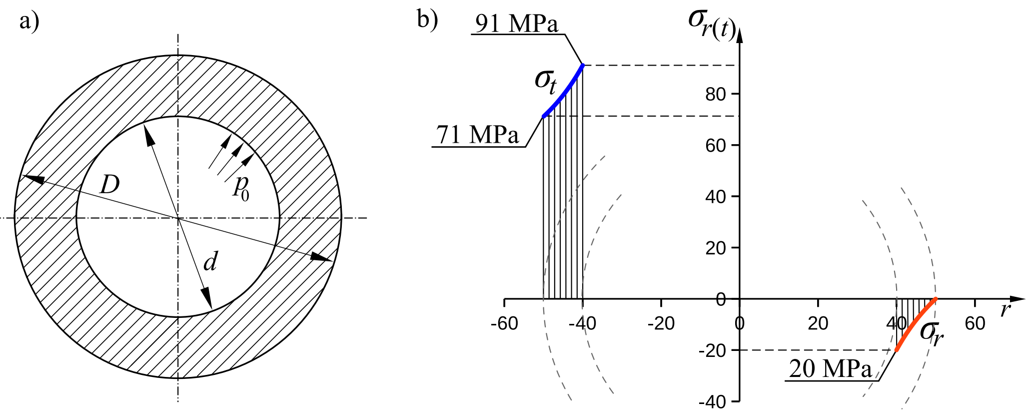

Fig. 4. Example 1: cross-section of the cylinder a); stress distribution b)

According to the general solution, equation (7), the following parameters can be defined $a = d/2$, $b = D/2$, $p_a = p_0$, $p_b = 0$. Substituting them into the solution one will obtain

$$\sigma_r = \frac{p_0 d^2}{D^2 - d^2}\left(1 - \frac{D^2}{4r^2}\right) \qquad \sigma_t = \frac{p_0 d^2}{D^2 - d^2}\left(1 + \frac{D^2}{4r^2}\right)$$

For the numbers provided in the content the plots shown in Fig. 4b are obtained. It is seen that the strength of the cylinder is influenced mainly by the hoop stress. Usually these stresses lead to the failure in the form of fracture which spread along the meridian of the cylinder.

Example 2

Compare the stress distribution in two variants of thick-cylinder loaded with internal pressure $5p$. The firs one (Fig. a) is made as a solid tube. The second one (Fig. b) is composed of two tubes connected with a shrink-fit. It is assumed that the pressure between tubes due to the shrink-fit equals p.

Let's start with the analysis of stress in the cylinder made with a single tube. The dimensions are: the external diameter a and the internal diameter $3a$. The cylinder is loaded only with the internal pressure only thus $\sigma_a = 5p$ and $\sigma_b = 0$. Substituting this numbers into the general solution, equation (7), we will get the formulae for determining the stress distribution

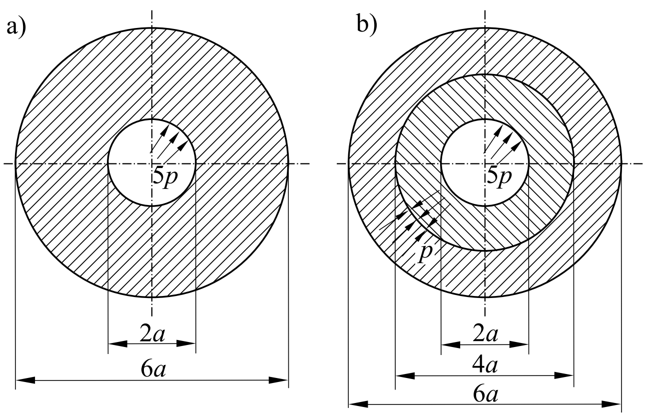

Fig. 5. Example 2: cross-section of the single-wall a) and double-wall b) cylinder

$$\sigma_r = \frac{5}{8}p\left(1 - \frac{9a^2}{r^2}\right)$$

$$\sigma_t = \frac{5}{8}p\left(1 + \frac{9a^2}{r^2}\right)$$

As can be seen if the cylinder is loaded only from one side the formulae simplify significantly. Since the plot is a power function and the cylinder is thick to obtain a smooth function flow we will determine its values in five points: $r_1 = a$, $r_2 = 1.5a$, $r_3 = 2a$, $r_4 = 2.5a$, $r_5 = 3a$. The obtained results are shown in Fig. 6a. Extreme values of the radial stress corresponds to the values of the pressure acting at particular faces. The stress are negative in the whole range.

The plot of the hoop stress looks similar, however, in this case the stress are positive in the whole cross-section and the value on the external face is different from zero. The maximum value of stress equals $6p$ and corresponds to the hoop stress on the inside face of the cylinder.

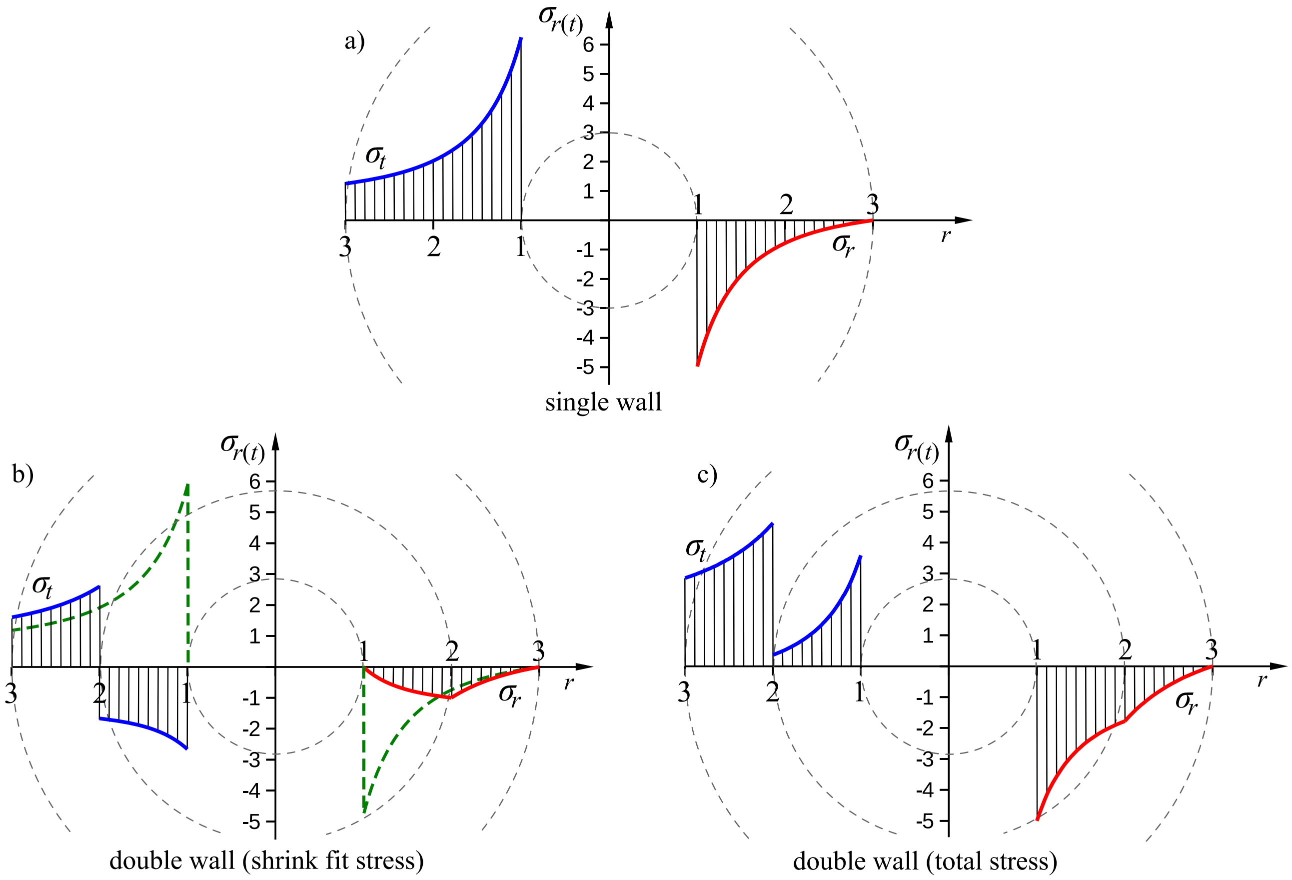

Fig. 6. Stress distribution in the thick-walled cylinder: single wall – loaded with internal pressure a); double wall – loaded with the shrink fit stress b); double wall – loaded with the shrink fit stress and internal pressure c)

Now let's consider the case in which the cylinder is composed of two tubes connected together with the shrink fit. There are two loads in this case – the pressure from the shrink fit acting at the interface of the two tubes and the internal pressure which is the working load. The solution can be obtained by dividing the task into two. In the first step the stress caused by the shrink fit will be determined only whereas in the second step only the stress due to internal pressure will be taken into account.

To determine the stress due the shrink fit the two tubes of the cylinder have to be considered separately according to Fig. 7. The internal tube will be loaded withe the external pressure. According to the assumptions made before we will have: $a = a$, $b = 2a$, $p_a = 0$ i $p_b = p$. Substituting this into the equations (7) there will be

$$\sigma_r = -\frac{4}{3}p\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_t = -\frac{4}{3}p\left(1 + \frac{a^2}{r^2}\right)$$

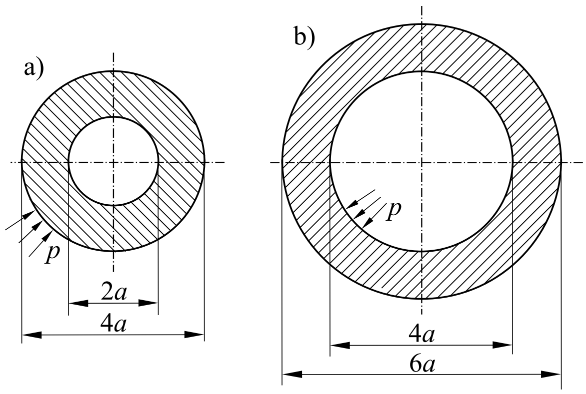

Fig. 7. The load of double wall cylinder

Stress will be determined for three values of the radius: $r_1 = a$, $r_2 = 1.5a$ and $r_3 = 2a$.

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In a similar way the stress for the external tube loaded with the internal load will be determined. Here we have: $a = 2a$, $b = 3a$, $p_a = p$ i $p_b = 0$ and the formulae for the stress takes the form

$$\sigma_r = \frac{4}{5}p\left(1 - \frac{9a^2}{r^2}\right)$$

$$\sigma_t = \frac{4}{5}p\left(1 + \frac{9a^2}{r^2}\right)$$

The values of the stress will be determined for the radii: $r_3 = 2a$, $r_4 = 2.5a$ and $r_5 = 3a$. Distribution of stress components are shown in Fig. 6b.

The radial stress reaches the maximum value equal to p at the interface of the two tubes whereas at the free faces equals zero. The distribution of stress is continuous. As to the hoop stress, at the interface there is a discontinuity and the stress change its sign. On the same plot the stress distribution for the cylinder loaded with the internal load only is shown with the dashed line. It has the same shape as the the one for the single wall cylinder. To obtain the final solution it is enough to add up the stress components for both loads – the pressure from the shrink fit and the one from the working load. It is shown in Fig. 6c.

To compare the two solutions it is enough to compare the plots shown in Fig. 6a and c. The most important goal that we managed to achieved is the decrease of the stress on the inside face which are now equal to 3.58, compare to $6p$ for the original design. It is especially important in the case when the load is applied suddenly – impact load. The maximum hoop stress are now equal to $4.63p$ and appears on the interface of the two tubes. The decrease of the maximum hoop stress is important because they are a tensile stress which may lead to the burst of the pipe. An additional advantage of the second solution is that the stress are more equally distributed on the thickness of the cylinder.

References

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