Axially symmetrical deformation
Rotating disks

One of the typical element used widely in the machine design is a rotating disk. Examples of such elements are flywheels, pulleys, brake disk or turbine rotors.

The strength of rotating disks is analysed similar to the strength of thick-walled cylinders. The difference is that in this case the disk has a finite thickness which has to be taken into account during analysis. Moreover, the thickness can vary along the radius of the disk. Also the load is different; there is no pressure applied to the disk but the load is generated by the inertia forces from routing mass

We take the following assumptions

- the disk is made of homogeneous material following Hooke's law,
- the disk is axially symmetrical and the symmetry plain is perpendicular to the axis of revolution,
- the thickness of the disk is relatively small.

Moreover we assume that the disk is in the plain state of stress the component of which are radial stress σ_r and hoop stress σ_t. Both components are functions of the radius r and angular velocity ω.

Five forces will be acting on the element abcd. Four of them are the forces coming from the stresses and these can be obtained by multiplying stress by the area they are working on. The fifth force is an inertia force F_b which is the result of centripetal acceleration a_n. This force

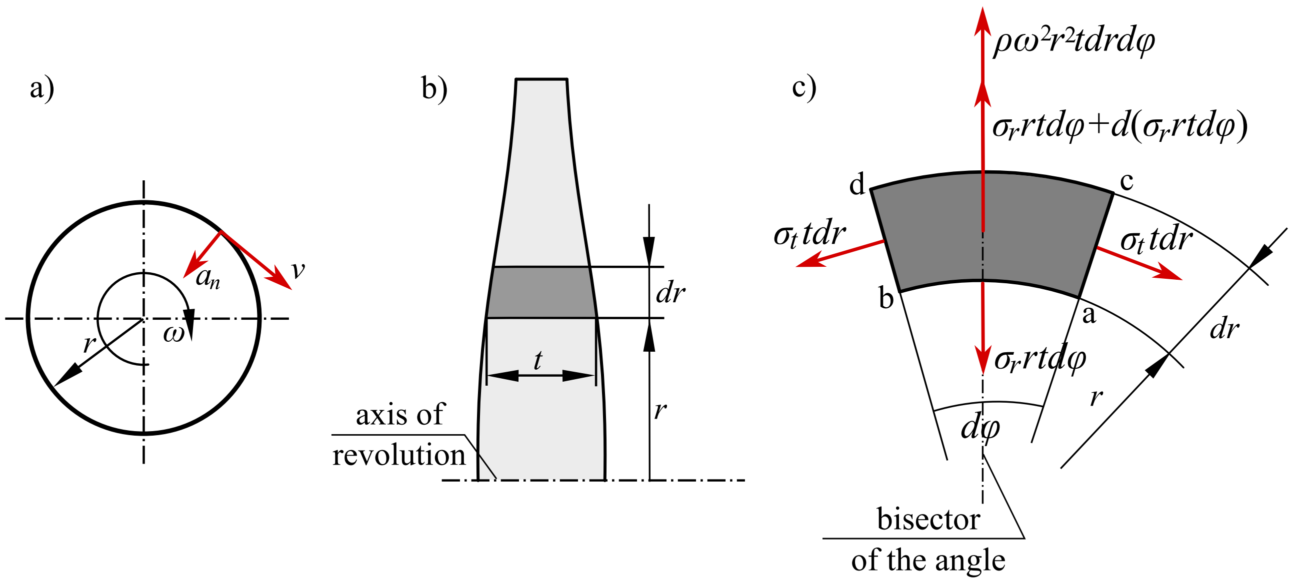

Fig. 1. Rotating disk: a) given parameters; b) radial section c) force distribution

will be equal to a product of the mass and acceleration $F_b = ma_n$. The centripetal acceleration is defined as $a_n = \omega^2 r$ whereas the mass is a product of the density ρ and the volume of the element abcd. Since the element is small, it has the length dr, it can be assumed that on this distance its thickness is constant. For small dimensions it will be truth to assume that the element has the shape of rectangular cuboid thus the volume will be equal to $V = rd\varphi drt$. The inertia force can be written as

$$F_d = \rho\omega^2 r^2 trdrd\varphi$$

The analysis will be carried out similar like in the case of thick-walled cylinder. The equation of equilibrium will be obtained by projecting all the forces on the bisector of the angle

$d\varphi$ and additional information will be received from the deformation analysis of the disk. The forces on the side faces are equal to each other whereas the force on the external face is bigger from this on the internal one about the infinitely increment. Thus, according to Fig. 1 there is

$$-\sigma_r trd\varphi + \sigma_r trd\varphi + d(\sigma_r trd\varphi) + \rho\omega^2 r^2 tdrd\varphi - \sigma_t tdrd\varphi = 0$$

The first two component can be reduced and the remaining part, after dividing by $drd\varphi$ will take the form

$$\frac{d}{dr}(\sigma_r tr) - \sigma_t t + \rho\omega^2 r^2 t = 0$$

The first element can be written as

$$\frac{d}{dr}(\sigma_r tr) = \frac{d}{dr}(\sigma_r t)r + \sigma_r t\frac{dr}{dr}$$

Substituting the above to the previous equation and dividing all by t we have

$$\frac{r}{t}\frac{d}{dr}\left(\sigma_r t\right) + \sigma_r - \sigma_t + \rho\omega^2 r^2 = 0 \tag{1}$$

A single equation is obtained in which there are two variables σ_r i σ_t. According to Hooke's law both of them can be expressed through strains which in this case are functions of one variable, the radius r. The strains can be determined the same way as for the thick-walled cylinder. Thus the radial and hoop strains in the rotating disk can be written as

$$\varepsilon_r = \frac{du}{dr} \quad \text{and} \quad \varepsilon_t = \frac{u}{r} \tag{2}$$

For the plain state of stress the Hooke's law has the form

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_t\right)$$
$$\sigma_t = \frac{E}{1-\nu^2}\left(\varepsilon_t + \nu\varepsilon_r\right)$$

and after substituting the strains

$$\begin{aligned}\sigma_r &= \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)\\ \sigma_t &= \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)\end{aligned} \tag{3}$$

Before we introduce the above relations into the equation of equilibrium (1) we will determine the derivative which appears in it

$$\frac{d}{dr}\left(\sigma_r t\right) = t\frac{d\sigma_r}{dr} + \sigma_r\frac{dt}{dr} = t\frac{E}{1-\nu^2}\left(\frac{d^2u}{dr^2} + \frac{\nu}{r}\frac{du}{dr} - \nu\frac{u}{r^2}\right) + \frac{dt}{dr}\frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

Substituting the above and the relations obtained from the Hooke's law to the equation of equilibrium, after reordering and multiplying by $(Er)/(1-\nu^2)$ we will obtain

$$\frac{d^2u}{dr^2} + \left(\frac{1}{r} + \frac{1}{t}\frac{dt}{dr}\right)\frac{du}{dr} + \frac{1}{r}\left(\frac{\nu}{t}\frac{dt}{dr} - \frac{1}{r}\right)u = Ar \tag{4}$$

where A is the constant describing the material and the motion

$$A = -\frac{1-\nu^2}{E}\rho\omega^2$$

Since the obtained equation (4) has a general character it is convenient to solve it for selected, specific cases.

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One of the possible solution of equation (4) is that in which the thickness of the disk t is constant along the whole radius. Thus, the solution will be valid for disks of constant thickness. If assume that $t = const.$ the derivative dt/dr will be equal to zero. The equation (4) will take the form

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = Ar \tag{5}$$

It can be written in the equivalent form

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ur)\right] = Ar$$

After double integration and division, each time, by r we will obtain

$$u = \frac{r^3}{8}A + \frac{r}{2}C_1 + \frac{1}{r}C_2 \tag{6}$$

By substituting the obtained equation for the displacement u into the Hooke's law the expressions for stresses can be obtained

$$\begin{aligned}\sigma_r &= \frac{E}{1-\nu^2}\left(\frac{(3+\nu)}{8}Ar^2 + \frac{1+\nu}{2}C_1 - \frac{1-\nu}{r^2}C_2\right)\\ \sigma_t &= \frac{E}{1-\nu^2}\left(\frac{(3+\nu)}{8}Ar^2 + \frac{1+\nu}{2}C_1 + \frac{1-\nu}{r^2}C_2\right)\end{aligned} \tag{7}$$

We are assuming the following boundary conditions, according to Fig. 2: first, for $r = 0$ $u = 0$ which means that the axis of the disk do not displace; second for $r = R_0$ $\sigma_r = 0$ which means that the outer surface of the disk is free from radial stress. From the first condition we will get $C_2 = 0$ whereas from the second one

$$C_1 = \frac{3+\nu}{4(1+\nu)}\frac{1-\nu^2}{E}\rho\omega^2R_0^2$$

Introducing constants to the system (9) we obtain the expressions for determining the stress components

$$\begin{aligned}\sigma_r &= \frac{3+\nu}{8}\rho\omega^2\left(R_0^2 - r^2\right)\\ \sigma_t &= \frac{3+\nu}{8}\rho\omega^2\left(R_0^2 - \frac{1+3\nu}{3+\nu}r^2\right)\end{aligned} \tag{8}$$

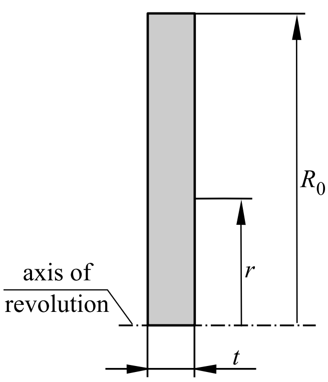

Fig. 2. Geometry of the disk

The above formulae allows to determine the stress along the radius of the disk. Theirs values depends on the rotational speed and the density of the material. They not depends on

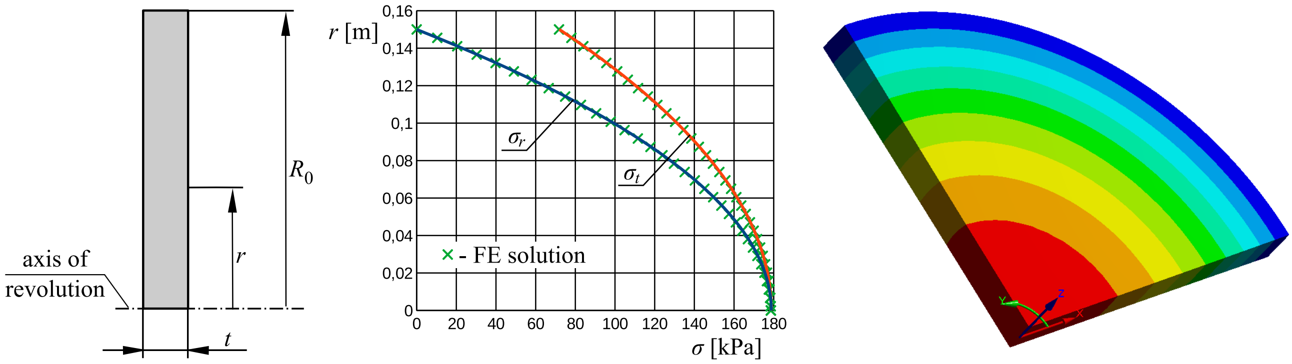

Fig. 3. Stress analysis in the rotating disk with constant thickness: a) geometry; b) stress distribution; c) view of the finite element solution

the thickness of the disk. An exemplary calculations are presented in Fig. 3 for the disk of the diameter 300 mm and the thickness 20 mm made of aluminium. The speed of rotation of the disk is 800 rpm. From the plot it is seen that at the axis of rotation both stress are equal to each other. On the free surface the radial stress equals zero and the hoop stress are different from zero. For comparison the results obtained from the finite element method are presented. The maximum discrepancy between the results obtained with both method is less than 0,5%.

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Another practical solution of the equation (4) can be obtained if we assume that $\sigma_r = \sigma_t = \sigma = const$. This way it is possible to obtain the disk of equal strength. At each point of the disk both components of stress has the same value.

Returning to the equation of equilibrium (1) having the form

$$\frac{r}{t}\frac{d}{dr}(\sigma_r t) + \sigma_r - \sigma_t + \rho\omega^2 r^2 = 0$$

and introducing the above assumption as to the stress we will get

$$\sigma\frac{r}{t}\frac{dt}{dr} + \rho\omega^2 r^2 = 0$$

To separate the variables we have to multiply the above equation by $dr/\sigma r$

$$\frac{dt}{t} = -\frac{\rho\omega^2}{\sigma}rdr$$

After integration we will get

$$\ln t = -\frac{\rho\omega^2 r^2}{2\sigma} + \ln C$$

The following steps are

$$\ln\frac{t}{C} = -\frac{\rho\omega^2 r^2}{2\sigma}$$

$$\log_e\frac{t}{C} = -\frac{\rho\omega^2 r^2}{2\sigma}$$

Remembering that $\log_a b = c \ \rightarrow \ a^c = b$ we obtain

$$t = Ce^{-\dfrac{\rho\omega^2 r^2}{2\sigma}}$$

The constant C can be determined from the boundary conditions. Let's assume that the thickness of the disk at the axis of rotation will have the thickness t_0. Thus, the boundary condition will be: for $r = 0$ $t = t_0$. From the above equation we will get $C = t_0$. The final formula will take the shape

$$t = t_0 e^{-\dfrac{\rho\omega^2 r^2}{2\sigma}} \tag{9}$$

With this formula it is possible to design of a rotating disk in which at each point both stress components have the same value. However it should be remembered that to fulfil this condition at the free end of the dicks the radial stress of the value σ must be applied.

An exemplary calculations have been conducted for an aluminium disk of the diameter equal to 300 mm, for which the thickness at the rotational axis equals $t_0 = 20$ mm. It was assumed that the stresses have to be equal to 10 MPa. Four different rotational speed has been selected to show how it influence the profile of the disk what is shown in Fig. 4. It is seen that

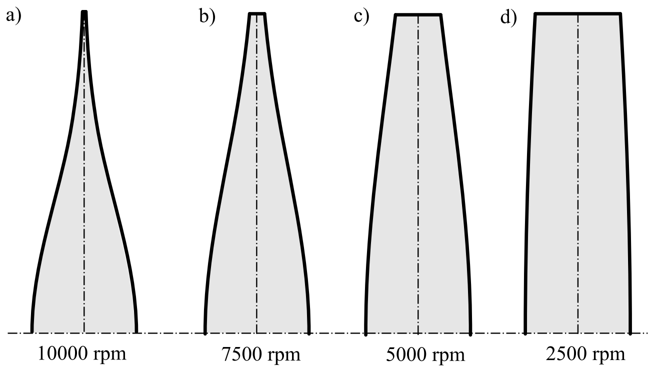

Fig. 4. Stress analysis in the rotating disk with constant strength: a) geometry; b) stress distribution; c) view of the finite element solution

with the increase of the rotational speed the thickness of the disk decreases with increasing of the radius. The correctness of the formulae can be verified by performing FE analysis of obtained shapes of the disk. The shape corresponding to the rotational velocity 10000 rpm has been selected and the results are shown in Fig. 5. Only small deviations can be observed

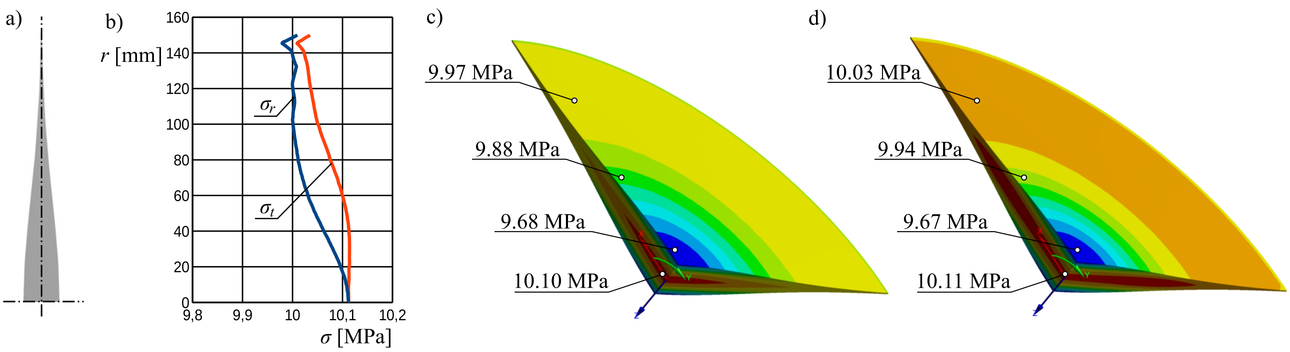

Fig. 5. Stress analysis in the rotating disk: a) geometry; b) stress distribution in the symmetry plane; c) radial stress; d) hoop stress

from the assumed stress value which was 10 MPa. It can be explained with a big difference of the thickness along the radius and additional deformation which take place along the axis of revolution – due to inertia forces the thickness of the disk tends to decrease.

References

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