Energy methods in mechanics – stability of structures

Example 1

The bar of the length l is pin-supported at one end and loaded with an axial force at the other end. At loaded point the bar may move in axial direction. Determine the critical load.

Solution

To solve this kind of problem with energy methods one must assume the shape function that is the shape the structure will take after the loss of stability. The solution which will be obtained is valid only for the assumed shape. In this case one can assume that at the moment of the loss of stability the end of the bar will move about u and the whole bar will bend in the way that in the mid-length the maximum deflection is v_{max}.

Thus, let's assume the shape function in the form of trigonometric function

$$v(x) = v_{max} \sin \frac{\pi x}{l}$$

The assumed function must fulfil the boundary conditions. In this case the conditions will have the following form: $v(0) = v(l) = 0$ and $v(\frac{l}{2}) = v_{max}$. It is seen that the assumed function will fulfil them.

The principle of stationary total potential energy has the form

$$\delta(U_\varepsilon - W) = 0$$

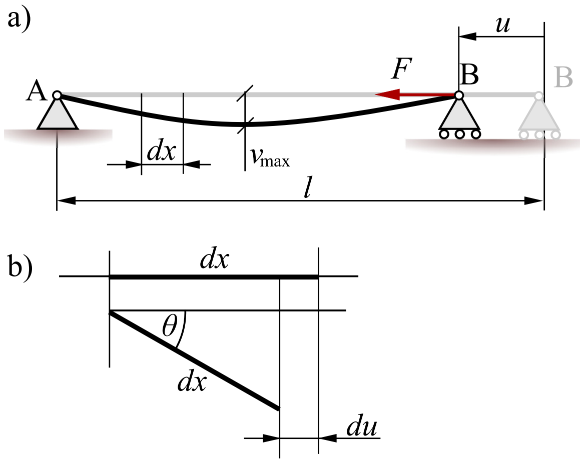

Fig. 1: Example 1

Thus, we must define first the energy of elastic deformation U_ε and second the work of external forces W.

We know that in the case of bending the energy of elastic deformation is expressed by the formula

$$U_\varepsilon = \frac{1}{2} \int_0^l \frac{M(x)^2}{EI} dx$$

We also know that the differential equation of the deflection curve of a beam has the form

$$\frac{d^2 v}{dx^2} = \frac{M(x)}{EI}$$

from which we have

$$M^2(x) = (EI)^2 \left(\frac{d^2 v}{dx^2} \right)^2$$

Substituting above to the energy equation there is

$$U_\varepsilon = \frac{EI}{2} \int_0^l \left(\frac{d^2 v}{dx^2} \right)^2 dx$$

This relation is the function of the assumed shape function $v(x)$.

Let's now define the work of external forces. From the definition, the work will equal the force F times the displacement u of the end of the bar.

$$W = Fu$$

One must notice that in the above relation there is no 0.5 factor. It appears only if the load is applied to the structure gradually from 0 to its maximum value F. In this case we define the energy which appears at the moment of the loss of stability and at this very moment the value of the load equals F – the maximum load is already applied.

We start with definition of the displacement u as a function of the shape function which can be done by analysing the elementary length of the beam dx. According to Fig. 1b a line segment dx after deformation equals the sum of the projection of dx on the axis x and an infinitely small displacement of the right end du. Thus we have

$$du = dx - dx\cos\theta = (1-\cos\theta)dx$$

Total displacement u is obtained by integration of the above expression along the length of the beam

$$u = \int\limits_l du = \int\limits_l (1-\cos\theta)dx$$

From trigonometric identities we know that $(1-\cos\theta) = 2\sin^2\frac{\theta}{2}$. Besides, for small deflections the angle of rotation of the beam is also small, thus $\sin\theta \approx \theta$. Having this in mind we have

$$u = \int\limits_l 2\sin^2\frac{\theta}{2}dx = \int\limits_l 2\left(\frac{\theta}{2}\right)^2 dx = \frac{1}{2}\int\limits_l \theta^2 dx$$

From bending theory of beams we know that the angle θ is the angle of rotation of the beam at a point and it is related with the deflection by the differential relation

$$\theta = \frac{dv}{dx}$$

Substituting the differential relation for the angle of rotation we have

$$u = \frac{1}{2}\int\limits_0^l \left(\frac{dv}{dx}\right)^2 dx$$

This equation allows to determine the displacement of the end of the beam if the shape function $v(x)$ is given.

Returning to the principle of stationary total potential energy

$$\delta(U_\varepsilon - W) = \delta\left[\frac{EI}{2}\int\limits_0^l \left(\frac{d^2v}{dx^2}\right)^2 dx - F\frac{1}{2}\int\limits_0^l \left(\frac{dv}{dx}\right)^2 dx\right] = 0$$

It is seen that under the integral sign there are derivatives of the shape function which in this case will take the form

$$\frac{dv}{dx} = v_{max}\frac{\pi}{l}\cos\frac{\pi x}{l}$$

$$\frac{d^2v}{dx^2} = -v_{max}\left(\frac{\pi}{l}\right)^2\sin\frac{\pi x}{l}$$

Substituting the integrals we will get

$$\delta\left[v_{max}^2\frac{EI}{2}\left(\frac{\pi}{l}\right)^4\int\limits_0^l \sin^2\frac{\pi x}{l}dx - v_{max}^2\frac{F}{2}\left(\frac{\pi}{l}\right)^2\int\limits_0^l \cos^2\frac{\pi x}{l}dx\right] = 0$$

Constant factors can be put in front of the variation sign and reduced. Both integrals calculated in the limits from 0 to l will give $l/2$. For the above equation to be truth the expression in the brackets must be equal to zero. Thus we have

$$EI\left(\frac{\pi}{l}\right)^2\frac{l}{2} - F\frac{l}{2} = 0$$

Finally the formula for the critical load is obtained.

$$F_{cr} = \frac{\pi^2 EI}{l^2}$$

It is identical with the one proposed by Euler.

Example 2 [Magnucki and Szyc, 2000]

A rigid column OC loaded with the force F is pin-connected with an elastic bar AB. Determine the critical load for the system.

Solution

In this case we can assume that the loss of stability will take place if the rigid column will rotate and the bar AB will undergo shortening. Thus the energy of elastic deformation will be the energy related with the shortening of the bar AB about Δl and the work of external load will be equal to the force F times the displacement of this force in the vertical direction.

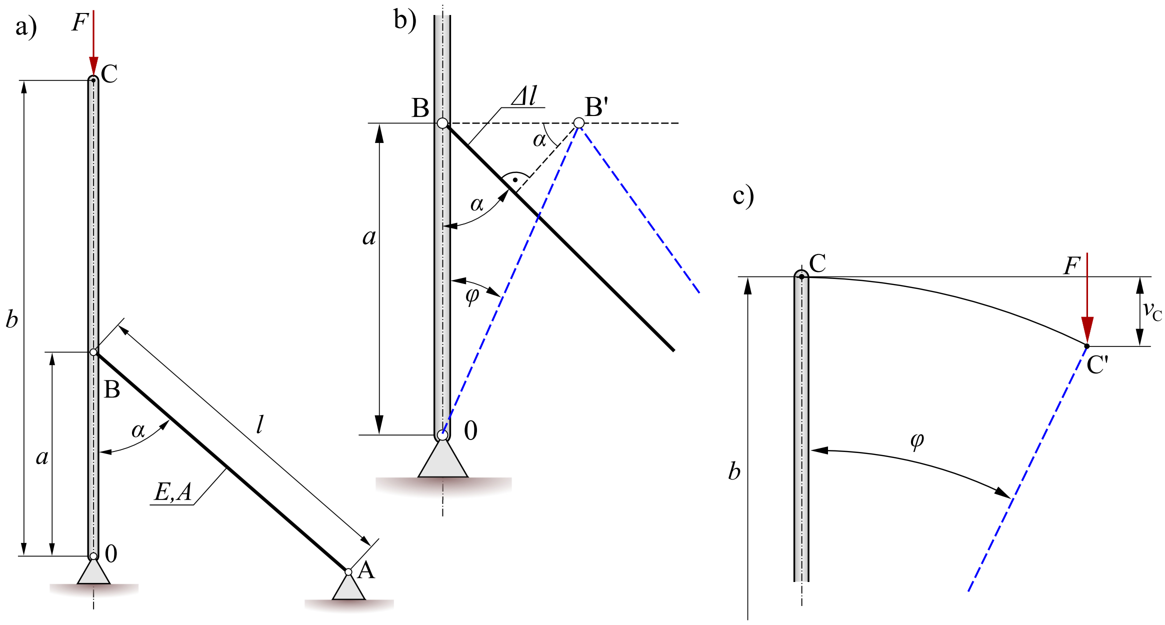

Fig. 2: Example 2

Since we are interested in the moment of the loss of stability, that is the very moment at which the bar starts to deform, its shortening will be very small. The change of the geometry of the system will be small too. Thus, we may assume that point B will translate along the line perpendicular to the axis of the column (the assumption of small deformations). According to Fig. 2b

$$\Delta l = \sin\alpha \text{BB'} \qquad \text{and} \qquad \text{BB'} = a\tan\varphi$$

Since the angle φ is small, we can approximately write $\tan\varphi \approx \varphi$ Finally, the shortening equals

$$\Delta l = a\varphi\sin\alpha$$

Now we can define the energy of elastic deformation, which for the bar under compression is expressed by the relation

$$U_\varepsilon = \frac{EA(\Delta l)^2}{2l}$$

Substituting the shortening Δl defined above we well get

$$U_\varepsilon = \frac{EA}{2l} a^2 \varphi^2 \sin^2 \alpha$$

Let's define the work of external load. According to Fig. 2c there is

$$v_c = b(1 - \cos\varphi)$$

From trigonometric identities we know that $(1 - \cos\varphi) = 2\sin^2\frac{\varphi}{2}$. Moreover, since the angle φ is small we have $\sin^2\frac{\varphi}{2} \approx (\frac{\varphi}{2})^2$. Finally the expression has the form

$$v_c = \frac{1}{2} b \varphi^2$$

and the work of external load

$$W = F v_c = \frac{1}{2} F b \varphi^2$$

The total potential energy of the system has the form

$$V = U_\varepsilon - W = \frac{EA}{2l} a^2 \varphi^2 \sin^2 \alpha - \frac{1}{2} F b \varphi^2$$

The only variable or in other words the only degree of freedom of the system is φ. The system will reach the critical state if the following conditions are fulfilled

$$\frac{\partial V}{\partial \varphi} = \frac{\partial^2 V}{\partial \varphi^2} = 0$$

Differentiating twice the above expression we obtain the solution in the form

$$F_{cr} = EA \frac{a^2}{bl} \sin^2 \alpha$$

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