Strength of Mechanical Constructions

Axially symmetrical deformation
Rotating disks

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Applications

Typical applications of rotating disks are

- flywheels
- brake disks
- pulleys
- turbine rotors

Assumptions for analysis

The strength of rotating disks is analysed similar to the strength of thick-walled cylinders. The difference is that:

- in this case the disk has a finite thickness which has to be taken into account during analysis
- the thickness can vary along the radius of the disk
- the load is generated by the inertia forces from rotating mass

Assumptions for analysis

We take the following assumptions:

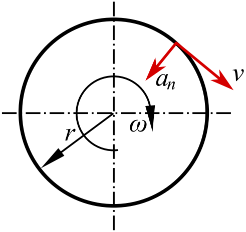
- the disk is made of homogeneous material following Hooke's law,
- the disk is axially symmetrical and the symmetry plain is perpendicular to the axis of revolution,
- the thickness of the disk is relatively small.

Moreover we assume that the disk is in the plain state of stress the component of which are radial stress σ_r and hoop stress σ_t. Both components are functions of the radius r and angular velocity ω.

Assumptions for analysis

Given parameters:

- radius of the disk r
- angular velocity ω
- centripetal acceleration a_n

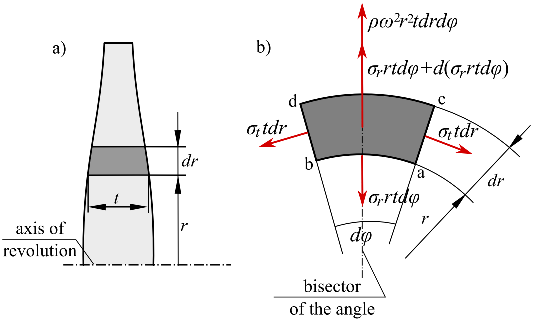


They are related with the following formula

$$a_n = \omega^2 r$$

Equilibrium of an element

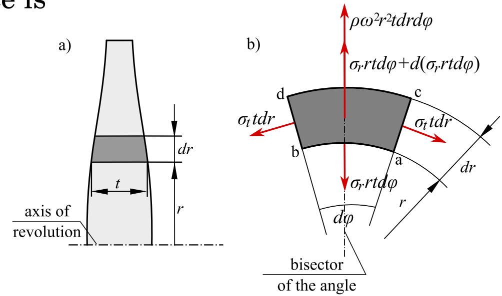
- analysis will be made on a small element of the thickness dr cut from the disk about r from the axis of rotation; it is limited by the angle $d\varphi$
- there are four forces acting of each face of the element



Equilibrium of an element

- additionally an inertia force F_b act on the element which is a result of the acceleration and mass $F_b = ma_n$
- acceleration is defined as $a_n = \omega^2 r$
- the mass is the product of the density ρ and the volume V which can be defined as $V = rd\varphi drt$
- thus the ineria force is

$$F_d = \rho\omega^2 r^2 trdrd\varphi$$



Equilibrium of an element

- equation of equilibrium will be obtained by projecting all the forces on the bisector of the angle $d\varphi$
- after reordering and dividing by $drd\varphi$ the equation will take the form

$$\frac{d}{dr}(\sigma_r t r) - \sigma_t t + \rho\omega^2 r^2 t = 0$$

- the first element can be written as

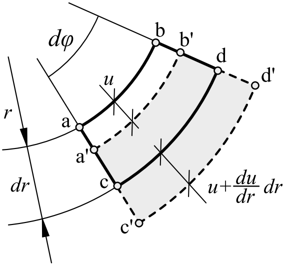
$$\frac{d}{dr}(\sigma_r t r) = \frac{d}{dr}(\sigma_r t)r + \sigma_r t \frac{dr}{dr}$$

- substituting the above to the previous equation and dividing all by t we have

$$\frac{r}{t}\frac{d}{dr}(\sigma_r t) + \sigma_r - \sigma_t + \rho\omega^2 r^2 = 0$$

Deformation of an element

- by analysing the equilibrium of the element only one equation of equilibrium is obtained containing two unknown functions:
  - radial stress σ_r
  - hoop stress σ_t
- to obtain additional equation the deformation of the element has to be considered



Hooke's law

- analysis of deformation gives

$$\varepsilon_r = \frac{du}{dr} \quad \text{and} \quad \varepsilon_t = \frac{u}{r}$$

- stresses and strains are related with the Hooke's law

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_t\right)$$

$$\sigma_t = \frac{E}{1-\nu^2}\left(\varepsilon_t + \nu\varepsilon_r\right)$$

Hooke's law

- this way both stress components become functions of the displacement u

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_t = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

- substituting the above into the equation of equilibrium gives the the differential equation with one variable which is u

Equation of equilibrium

- the final form of the equation of equilibrium has the form

$$\frac{d^2u}{dr^2} + \left(\frac{1}{r} + \frac{1}{t}\frac{dt}{dr}\right)\frac{du}{dr} + \frac{1}{r}\left(\frac{\nu}{t}\frac{dt}{dr} - \frac{1}{r}\right)u = Ar$$

  where A is the constant describing the material and the motion

$$A = -\frac{1-\nu^2}{E}\rho\omega^2$$

- since the obtained equation has a general character it is convenient to solve it for selected, specific cases.

Solid disk with constant thickness

- one of the possible solution of equation of equilibrium is that in which the thickness of the disk t is constant along the whole radius
- if it is assumed that $t = const.$ the derivative dt/dr will be equal to zero; the equation will take the form

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = Ar$$

or

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ur)\right] = Ar$$

Solid disk with constant thickness

- the solution has the form

$$u = \frac{r^3}{8}A + \frac{r}{2}C_1 + \frac{1}{r}C_2$$

- and the stress components

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{(3+\nu)}{8}Ar^2 + \frac{1+\nu}{2}C_1 - \frac{1-\nu}{r^2}C_2\right)$$
$$\sigma_t = \frac{E}{1-\nu^2}\left(\frac{(3+\nu)}{8}Ar^2 + \frac{1+\nu}{2}C_1 + \frac{1-\nu}{r^2}C_2\right)$$

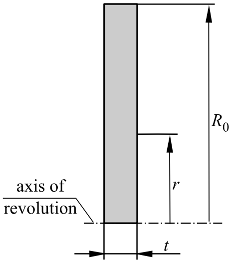
Solid disk with constant thickness

The following boundary conditions are assumed

- for $r = 0$ $u = 0$ which means that the axis of the disk do not displace
- for $r = R_0$ $\sigma_r = 0$ which means that the outer surface of the disk is free from radial stress

The constants will take the value

- $C_2 = 0$
- $C_1 = \dfrac{3+\nu}{4(1+\nu)} \dfrac{1-\nu^2}{E} \rho\omega^2 R_0^2$



Solid disk with constant thickness

The stress components will take the form

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(R_0^2 - r^2\right)$$

$$\sigma_t = \frac{3+\nu}{8}\rho\omega^2\left(R_0^2 - \frac{1+3\nu}{3+\nu}r^2\right)$$

The above formulae allows to determine the stress along the radius of the disk. Theirs values depends on the rotational speed and the density of the material. They not depends on the thickness of the disk.

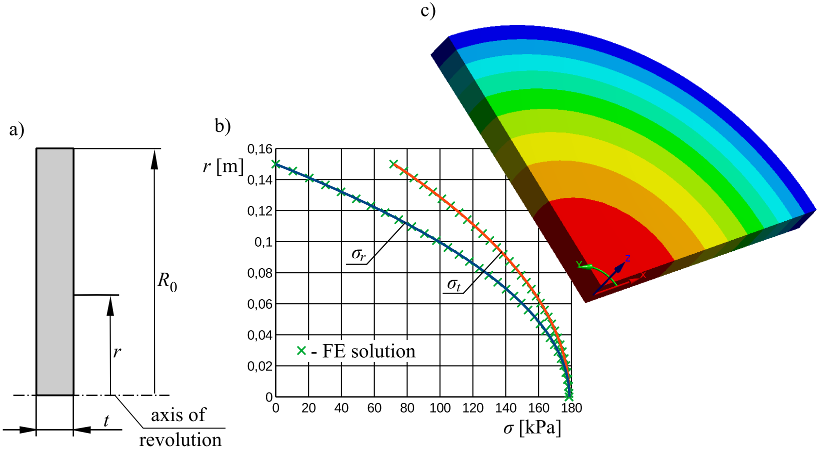
Solid disk with constant thickness

Example

Determine the radial and hoop stress for an aluminium disk of the diameter equal to 300 mm, and thickness equal to 20 mm. The speed of rotation of disk is 800 rpm.

Solid disk with constant thickness

Example – solution



Solid disk of constant strength

- another practical solution of the equation of equilibrium can be obtained if we assume that $\sigma_r = \sigma_t = \sigma = const.$
- this way it is possible to obtain the disk of equal strength – at each point of the disk both components of stress has the same value.
- equation of equilibrium has the form

$$\frac{r}{t}\frac{d}{dr}(\sigma_r t) + \sigma_r - \sigma_t + \rho\omega^2 r^2 = 0$$

- assuming equal stress we have

$$\sigma\frac{r}{t}\frac{dt}{dr} + \rho\omega^2 r^2 = 0$$

Solid disk of constant strength

- solution of this equation for t gives

$$t = Ce^{-\dfrac{\rho\omega^2 r^2}{2\sigma}}$$

- the constant C can be determined from the boundary conditions
- let's assume that the thickness of the disk at the axis of rotation will have the thickness t_0
- the boundary condition will be: for $r = 0$ $t = t_0$
- from the above equation we will get $C = t_0$ and the final formula will take the shape

$$t = t_0 e^{-\dfrac{\rho\omega^2 r^2}{2\sigma}}$$

Solid disk of constant strength

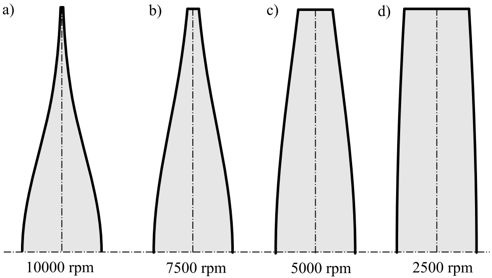
Example

Determine the shape of the solid rotating disk made of aluminium of the diameter 300 mm and the thickness at the rotational axis equal to 20 mm. Assume the value of both stress components equal to 10 MPa. Perform calculations for four different rotational speed: 2500, 5000, 7500 and 10000 rpm.

Solid disk of constant strength

Example – solution

Cross-sections of disks of equal strength for different rotational speed



Solid disk of constant strength

Example – solution

FE solution for 10000 rpm