Strength of Mechanical Constructions

Axially symmetrical deformation
Thick-walled cylinders

Paweł JASION, PhD. Eng.
e-mail: pawel.jasion@put.poznan.pl
www: pawel.jasion.pracownik.put.poznan.pl

Poznan University of Technology
Institute of Applied Mechanics
Division of Strength of Materials and Structures

Applications

Typical applications of thick-walled cylinders are

- thick-walled pressure vessels
- hydraulic cylinders
- gun barrel
- pipes carrying fluid at high temperature
- press fit or shrink fit

Assumptions for analysis

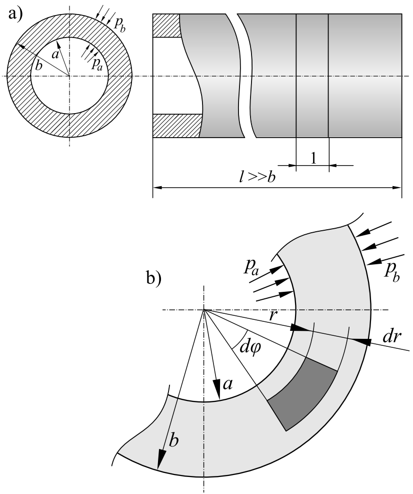
Thick-walled cylinders are usually loaded with axially symmetrical load in the form of

- internal or/and external pressure
- high temperature

If the cylinder is not closed, long and the stress analysis is made far from the support the following assumptions are valid:

- deformation is symmetrical about the axis and do not change along the length
- due to symmetry there is no shearing forces
- plain sections remain plain after the load is applied

Geometry of thick-cylinder

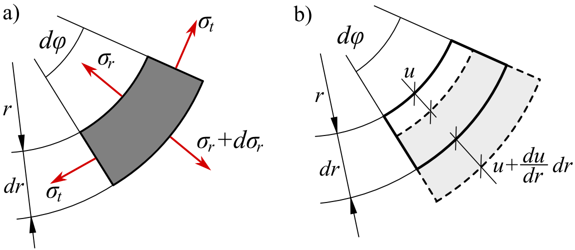


- radii of the cylinder
  - internal a
  - external b
- loading pressure
  - internal p_a
  - external p_b

Derivation of stress components

Lame's equations

- by analysing the equilibrium of the element only one equation of equilibrium is obtained containing two unknown functions:
  - radial stress σ_r
  - hoop stress σ_t
- to obtain additional equation the deformation of the element has to be considered



Derivation of stress components

Lame's equations

- from equation of eqiulibrium the following equation is obtaitned

$$\sigma_r - \sigma_t + \frac{d\sigma_r}{dr} r = 0$$

- analylis of deformation gives

$$\varepsilon_r = \frac{du}{dr} \quad \text{and} \quad \varepsilon_t = \frac{u}{r}$$

- all above quantities are related with the Hooke's law

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_t\right)$$

$$\sigma_t = \frac{E}{1-\nu^2}\left(\varepsilon_t + \nu\varepsilon_r\right)$$

Derivation of stress components

Lame's equations

- this way both stress components become functions of the displacement u

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_t = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

- substituting the above into the equation of equilibrium gives the the differential equation with one variable which is u

Derivation of stress components

Lame's equations

- the differential equation is obtained in the form

$$\frac{d^2u}{dx^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

  the solution of which is

$$u = C_1 r - \frac{C_2}{r}$$

- constants C_1 and C_2 can be calculated from the boundary conditions at the internal and external surfaces of the cylinder

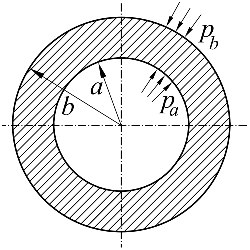
Derivation of stress components

Lame's equations

The general form of stress components depending on the boundary conditions

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\frac{1-\nu}{r^2}\right]$$

$$\sigma_t = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\frac{1-\nu}{r^2}\right]$$



The possible boundary conditions are:

- for $r = b \rightarrow \sigma_r = p_b$ and for $r = a \rightarrow \sigma_r = p_a$
- for $r = b \rightarrow \sigma_r = p_b$ and for $r = a \rightarrow \sigma_r = 0$
- for $r = b \rightarrow \sigma_r = 0$ and for $r = a \rightarrow \sigma_r = p_a$

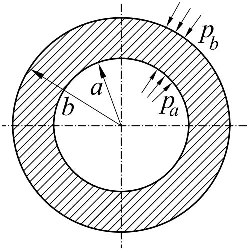
Derivation of stress components

Lame's equations

For the case when both pressures are acting the formulae for stress have the form

$$\sigma_r = \frac{p_a a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) - \frac{p_b b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)$$

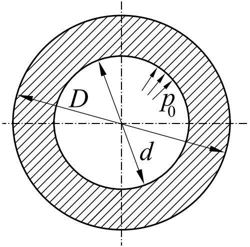
$$\sigma_t = \frac{p_a a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) - \frac{p_b b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$



Application of Lame's equations

Example 1

Determine the stress distribution in a thick-walled hydraulic cylinder for which the maximum working pressure is $p_0 = 20$ MPa. Internal diameter $d = 40$ mm and external diameter $D = 50$ mm. The cylinder is made of stainless steel with Young modulus $E = 200000$ MPa.



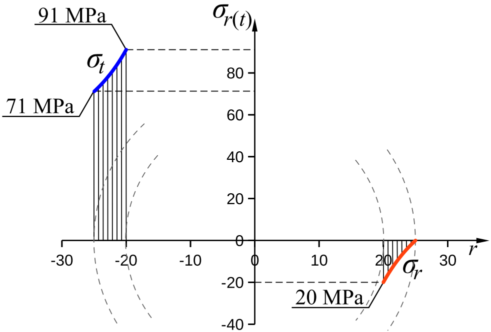
Application of Lame's equations

Example 1

Analytical solution

$$\sigma_r = \frac{p_a a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

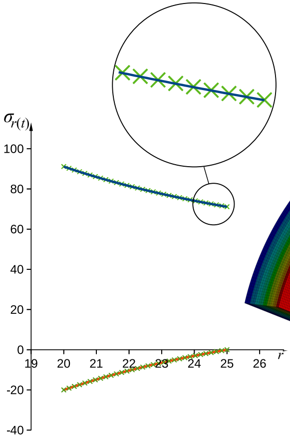
$$\sigma_t = \frac{p_a a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

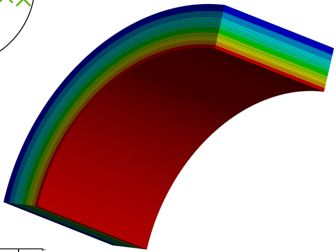


Application of Lame's equations

Example 1 – FEM solution

Comparison of analytical solution (solid lines) and FEM solution (crosses)





Application of Lame's equations

Analysis of equations

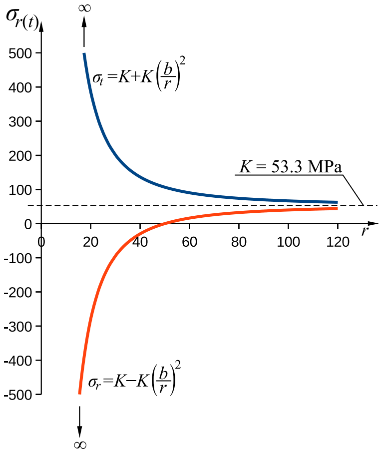
- for internal load we can write

$$\sigma_r = K - K\left(\frac{b}{r}\right)^2$$

$$\sigma_t = K + K\left(\frac{b}{r}\right)^2$$

where

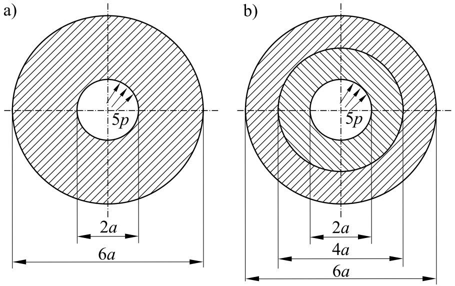
$$K = \frac{p_a a^2}{b^2 - a^2}$$



Application of Lame's equations

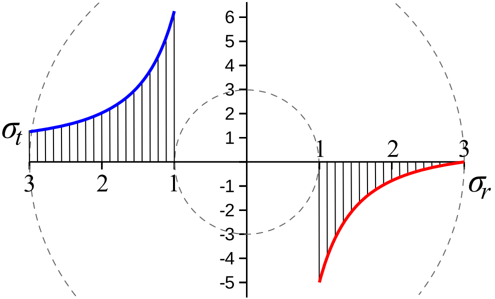
Example 2

Compare the stress distribution in two variants of thick-cylinder loaded with internal pressure $5p$. The firs one (Fig. a) is made as a solid tube. The second one (Fig. b) is composed of two tubes connected with a shrink-fit. It is assumed that the pressure between tubes due to the shrink-fit equals p.



Application of Lame's equations

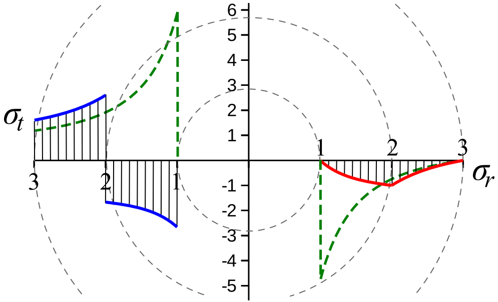
Example 2 – solution



solid tube

Application of Lame's equations

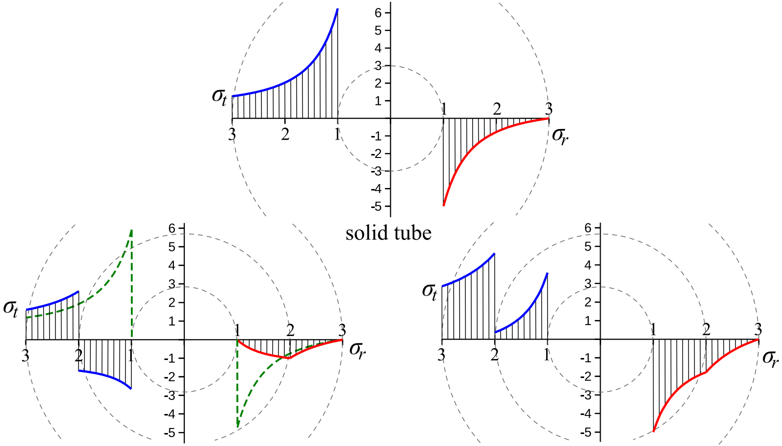
Example 2 – solution



composed tube (shrink-fit stress)

Application of Lame's equations

Example 2 – solution



solid tube

composed tube (shrink-fit stress)

composed tube (total stress)