Strength of Mechanical Constructions

Energy methods in mechanics
Stability of structures

Paweł JASION, PhD. Eng.
e-mail: pawel.jasion@put.poznan.pl
www: pawel.jasion@pracownik.put.poznan.pl

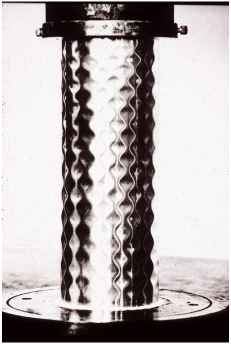
Poznan University of Technology
Institute of Applied Mechanics
Division of Strength of Materials and Structures

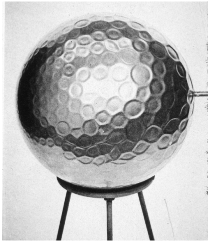
Stability of structures

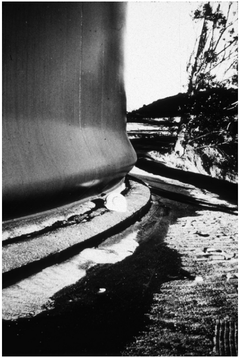
- thin-walled or slender structures may loss its stability due to compressing forces

Stability of structures

Examples of loss of stability

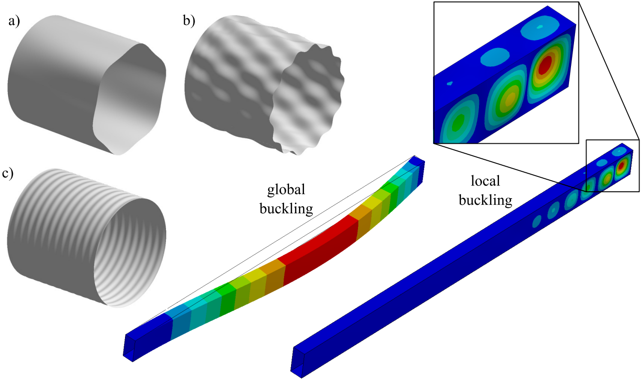






Stability of structures

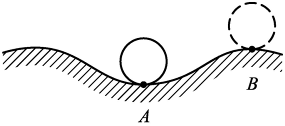
- buckling mode has usually the form of one or several waves
- the loss of stability may have global or local character



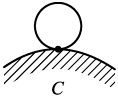
Types of stability

- in mechanics objects in equilibrium are analysed; it is not important what is the type of equilibrium – stability is not investigated
- stability is the ability of structure to maintain is state of stable equilibrium after external disturbance

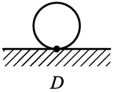
the equilibrium state is stable if any small disturbance cause a small deviation of the system from this state



stable equilibrium



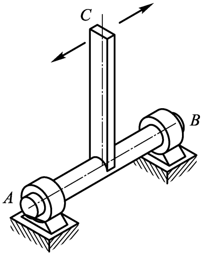
unstable equilibrium



neutral equilibrium

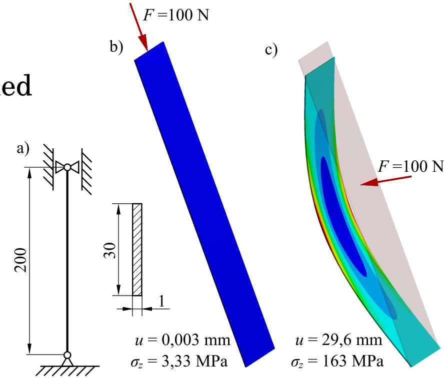
Types of stability

- the response of the system on disturbance of equilibrium state, that is the stability of the system, depends on:
  - type of structural element (column, plate, shell)
  - geometrical dimensions
  - type of load and support
  - initial geometrical imperfections
- thus, the stability analysis has to be made individually for a given system and different types of disturbances have to be taken into account (e.g. loads in different directions)



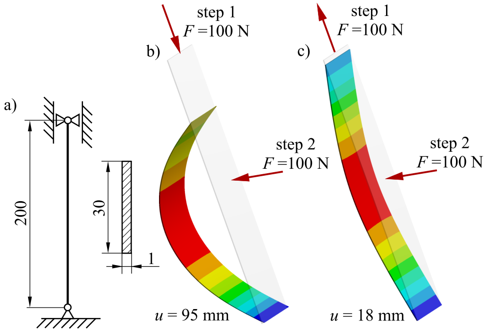
Membrane state of stress

- slim and thin-walled structures bare the load through the membrane forces
- deformations corresponding to this forces are very small
- but the deformation due to transverse forces are big
- thus slim and thin-walled structures have large membrane stiffness and small bending stiffness



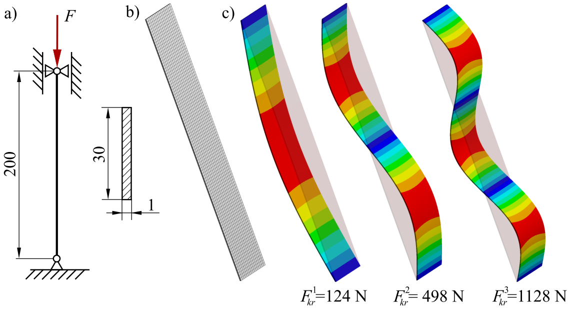
Membrane state of stress

- membrane forces influences the bending stiffness of the structure in two ways
  - compression forces decrease the stiffness
  - tensile forces increase the stiffness (*stress stiffening*)



Membrane state of stress

- in an extreme case compression membrane forces may be so big that the stiffness of the structure will drops to zero; then, after applying very small load (disturbance) or even without this load the structure bends – it loose its stability



Membrane state of stress

- the phenomenon of the loss of stability can be explained with the use of energy approach
- the load applied to the structure induce the displacement; then the work is done which is transferred into the energy of elastic deformation

$$W_e = \frac{1}{2} F \Delta l; \qquad U_\varepsilon = \frac{F^2 l}{2EA}$$

- since the membrane forces are big this energy is also big

Membrane state of stress

- at the moment of loss of stability the energy of deformation of the membrane state is transformed into the energy of bending state
- since the bending stiffness is much smaller a large deformation is necessary to absorb the energy
- the amount of energy stay on the same level but the geometrical configuration of the structure is different
- usually it leads to the destruction of the structure

Analysis of structures

The in-depth analysis of the behaviour of a structure covers three stages

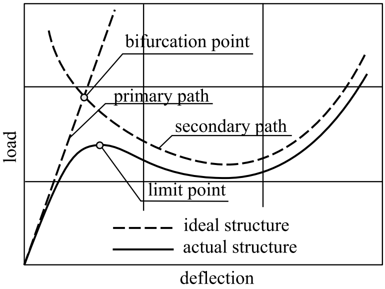
- pre-buckling state
  - displacements and stresses are determined
- critical state
  - buckling load and buckling shape are determined
- post-buckling state
  - the behaviour of the structure after the loss of stability is described; usually the failure appears here

Types of buckling

Equilibrium path

Equilibrium path

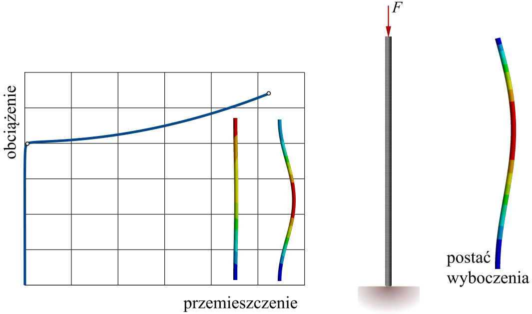
relation between the displacement of particular point of the structure and the load



Types of buckling
Equilibrium path

Different types of equilibrium paths can be obtained by analysing the following problems:

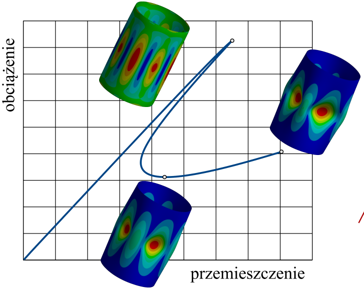
- axially compressed column
  - equilibrium path is stable; it means that after the critical point is exceeded a further deformation is possible only with the increase of the load

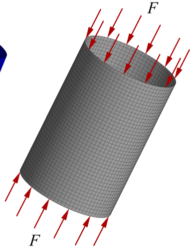


Types of buckling

Equilibrium path

- axially compressed cylindrical shell or spherical shell under external pressure
  - in both cases the path is not stable; after the critical point is exceeded a sudden and substantial drop of the load is observed; the structure collapse

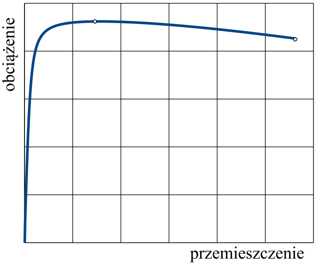


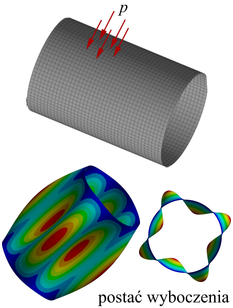


Types of buckling

Equilibrium path

- cylindrical shell under external pressure
  - equilibrium path is unstable; after exceeding the critical load any additional force is necessary to increase the deformation

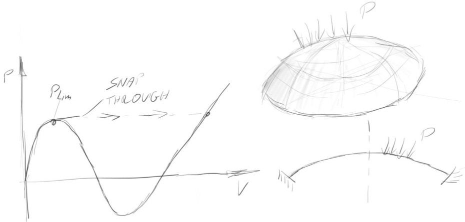




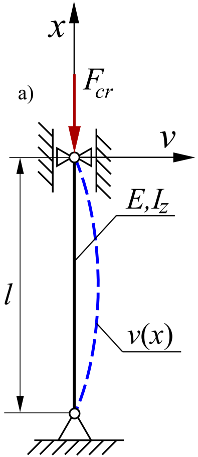
Types of buckling

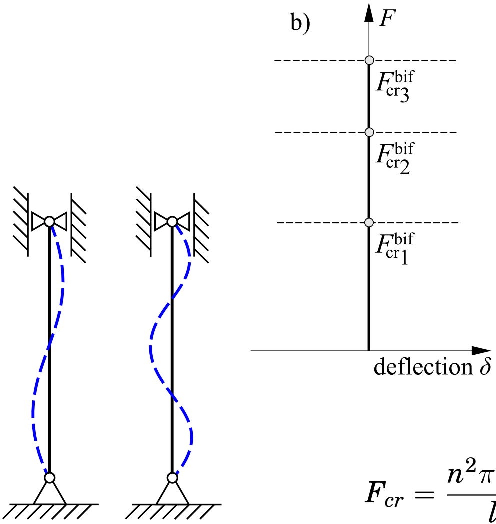
Equilibrium path

- spherical cap under external pressure
  - equilibrium path is unstable; a snap-through is observed – to retain the current deformation a negative force has to be applied



Stability of compressed column





$$F_{cr} = \frac{n^2\pi^2 EI_z}{l^2}$$

Principle of stationary total potential energy

Total potential energy of elastic system

$$V = U_\varepsilon - W$$

Lagrange-Dirichlet theorem

if the total potential has a relative minimum at an equilibrium position, then the equilibrium position is stable

Principle of stationary total potential energy

$$\delta(U_\varepsilon - W) = 0$$

Equilibrium of a mechanical system

- the principle of stationary total potential energy states that the first variation of the total potential energy has to be equal to zero $\delta(V) = 0$; this conditions is sufficient to determine the critical state – the state of equilibrium
- if it is necessary to determine the character of equilibrium and additional condition has to be checked
    - if $\delta^2(V) < 0$ then the equilibrium is unstable
    - if $\delta^2(V) > 0$, then the equilibrium is stable
    - if $\delta^2(V) = 0$, then the equilibrium is neutral

Energy approach to stability analysis

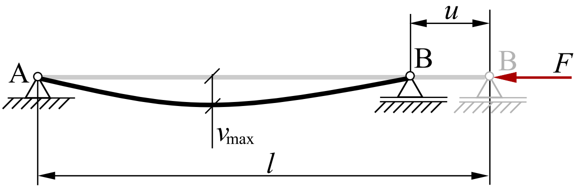
To solve the problem of stability using energy method, that is to derive the equation of equilibrium, the following steps should be defined:

- field of displacement: $u = u(x)$
- strain: $\varepsilon = du/dx$
- stress based on the Hooke's law: $\sigma = E\varepsilon$
- energy of elastic deformation: U_ε
- work of load: W

Stability of structures

Example 1

Determine the value of the critical load F for the axially loaded beam of the stiffness EI.



Equilibrium of a mechanical system

- the above conditions are proper if the energy is expressed in the form of functional; the argument of energy function is another function e.g. the shape function
- if the energy is a function of variables it is enough to determine the derivatives and the critical load can be calculated by checking the following conditions

$$\frac{\partial V}{\partial \delta} = 0 \qquad \text{and} \qquad \frac{\partial^2 V}{\partial \delta^2} = 0$$

where V is the total potential energy of the system and δ is the generalised coordinate (displacement, angle of rotation)

Stability of structures

Example 2

A rigid column OC loaded with the force F is pin-connected with an elastic bar AB. Determine the critical load for the system.