Stresses in curved bars

Stresses in curved bars are the results of acting of three internal forces: normal force N, shear force V and bending moment M. The calculation of the first two can be made in a regular way as in the case of a straigh bar. Thus we will focus on the third one which are normal stresses due to bending. Determination of the stress formula is a statically indeterminate problem which means that besides the equations of equilibrium the deformation of the bar needs to be considered. There are three elements that must be determined: the shape of the stress distribution on the cross-section, the location of the neutral line and the relation between the bending moment and the value of the stress. We will start our considerations from the analysis of deformation of an elementary part of the bar and then we will switch to the analysis of the equilibrium of a part of the bar.

Let's consider a curved bar in a pure bending conditions as show in the Fig. 1. From this bar, far from the point of application of the load, we will cut an element described by the angle $d\theta$. Before we start to analyse the deformation of the element it is necessary to make the following assumptions:

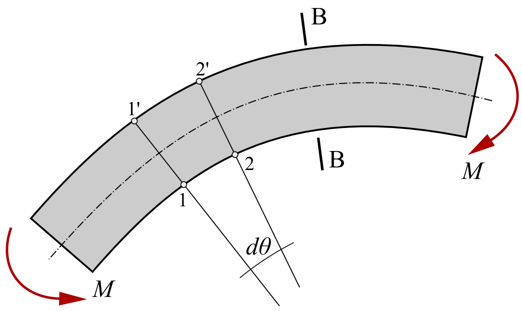

Fig. 1: Curved beam under pure bending

- transverse cross-section plane and normal to the neutral line remains so after deformation,
- rotation takes place according to the neutral line,
- elongation and contraction of fibres as well as stresses along lines of equal distance from the neutral line are the same.

The angle which describe the cut element of the bar equals $d\theta$ (see Fig. 2). Since the length of the element is small we can assume that its curvature is described with a circular arc. On the drawing we can distinguish a central line passing through the centroid of the cross-section and of the radius R_0 as well as the neutral line O_1O_2 for which the stress are equal to zero and the radius of which equals r. Since at this moment the position of the neutral line is not known it is assumed that it does not corresponds to the central line. Besides an arbitrary fibre at the radius ρ marked as A_1A_2 is selected. This fibre is at a distance y from the neutral line.

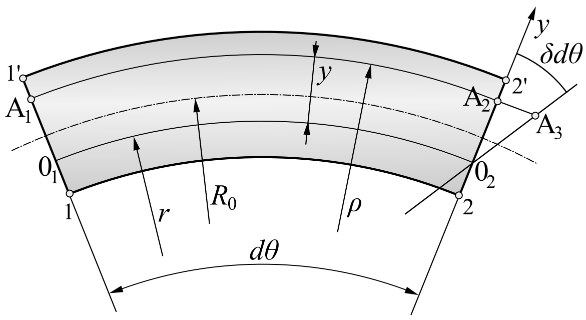

Fig. 2: Element of the beam

During the deformation the cross-section of the bar rotates about small angle $\delta d\theta$. To find the relation between this angle and the deformation of fibres of the bar the strains in the fibre A_1A_2 will be analysed. Initial length of the fibre equals $\rho d\theta$. The increase of the length of the fibre due to rotation of the cross-section is equal to $y\delta d\theta$. From definition we know that the strain of the element equals the ratio of the increase of the length to the initial length, that is $\varepsilon = \Delta l/l$. In this case we have

$$\varepsilon = \frac{A_2A_3}{A_1A_2} = \frac{y\delta d\theta}{\rho d\theta} \tag{1}$$

Applying the Hooke's law the distribution of stress on the cross-section of the curved beam is obtained in the form

$$\sigma = \varepsilon E = \frac{y}{\rho}\frac{\delta d\theta}{d\theta}E = \frac{\rho - r}{\rho}\frac{\delta d\theta}{d\theta}E \tag{2}$$

Both E and $(\delta d\theta)/(d\theta)$ are constant values, thus we can divide the formula (2) by them and obtain the dimensionless stress in the form

$$\tilde{\sigma} = 1 - \frac{r}{\rho} \tag{3}$$

which gives the possibility to visualise easily the distribution of stress. From this formula it is seen that it has a hyperbolic character. It is not the straight line like in the straight beams. From the plot shown in Fig. 3 it is seen that stresses increase rapidly below the neutral axis for small values of the radius.

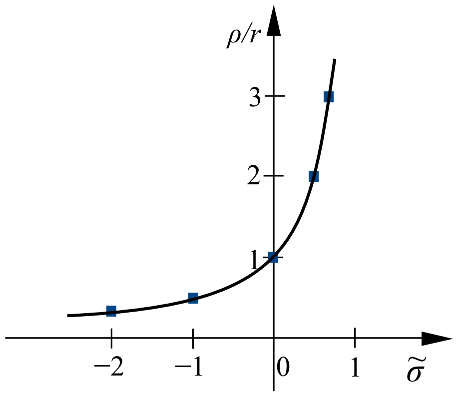

Fig. 3: Stress distribution

For now we now what the stress distribution is but we still do not know where the neutral axis is located and how the stress value is related to the bending moment. For this reason we will consider the equilibrium of a part of the beam cut at location B seen in Fig. 1. The left hand side of the beam will be considered.

There are six equations of equilibrium that can be written in this case. However, most of them will not give us any valuable information. According to Fig. 4 the only force acting on the cut part of the beam is the elementary normal force $dN = \sigma dA$. For that reason the only force equation will be that according to axis x

$$\sum F_x = 0 \;\rightarrow\; \int_A \sigma dA = 0. \tag{4}$$

As to equations of moments the force σdA is parallel to the axis x and for this reason it will not generate any moment. The same force generates moment according to axis y, however due to the symmetry of the problem and the assumption of the plane bending the total value will be equal to zero. The only equation which will help to find the stress formula is the equation of moments according to axis z. There are two components which will appear in this equation: the moment generated by the force σdA and the external load M

$$\sum M_z = 0 \;\rightarrow\; M - \int_A \sigma y dA = 0 \tag{5}$$

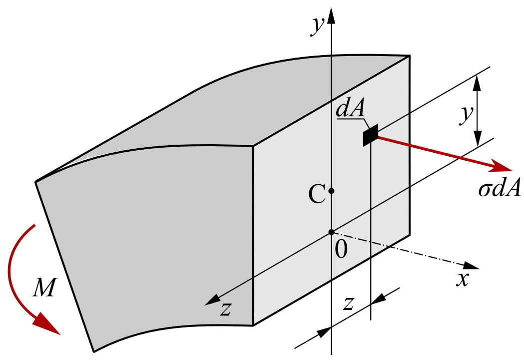

Fig. 4: Analysis of stress

These two equations contain the stress σ which were determined previously (2) from analysis of deformation of the beam. By substituting them into equations of equilibrium we can get the necessary information.

Let's start with the equation of forces (4). After putting constants outside the integral it will take the form

$$E\frac{\delta d\theta}{d\theta}\int_A \frac{y}{\rho} dA = 0 \tag{6}$$

Since the constants are different from zero it is the integral that has to be equal to zero, that is

$$\int_A \frac{y}{\rho} dA = 0 \tag{7}$$

Knowing, from Fig. 2, that $y = \rho - r$ we can write

$$\int\limits_A dA - \int\limits_A \frac{r}{\rho} dA = 0 \tag{8}$$

The first integral is the area of the cross-section A. Since the radius of the neutral line r is a constant value we can determine it from the above equation

$$r = \frac{A}{\displaystyle\int\limits_A \frac{dA}{\rho}} \tag{9}$$

Knowing the radius r we can determine the position of the neutral line which is $y_0 = R_0 - r$ and that was our first goal.

Now we may switch the the equation of equilibrium of moments (5) and substitute σ with (2). The result will be

$$M - E\frac{\delta d\theta}{d\theta} \int\limits_A \frac{y^2}{\rho} dA = 0 \tag{10}$$

Let's consider the integral alone. Knowing that $y = \rho - r$ we have

$$\int\limits_A \frac{y^2}{\rho} = \int\limits_A y dA - r \int\limits_A \frac{y}{\rho} dA \tag{11}$$

From previous analysis (see Eq. (7)) we know that the second integral equals zero. Moreover, the first equation is the static moment S. Than the equation of equilibrium takes the form

$$M - E\frac{\delta d\theta}{d\theta} S = 0 \tag{12}$$

When analysing the above equation it is seen that for loaded beam the bending moment M is different from zero. Both constants E and $(\delta d\theta)/(d\theta)$ are also different from zero. If we want this equation to be fulfilled the static moment S must be different from zero too. Since this moment equals zero for the centroidal axis it means that in the problem of bending of curved bars the centroidal axis do not coincide with the neutral axis; there is a shift equals y_0 with which we can calculate the static moment $S = Ay_0$. It should be noted that the shift of the two axes is a consequence of the stress distribution. Since the maximum stress on the outside fibres of the bar are much smaller that these on the inner side the area at which the former works has to be bigger. Or in other words to have the equilibrium at the cross-section the positive forces N^+ have to equalise the negative ones N^- as shown in Fig. 5.

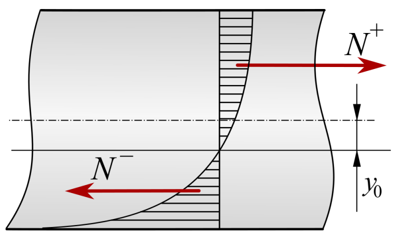

Fig. 5: Normal forces at the cross-section

To obtain the final formula for normal stress let's write the Eq. (12) in the form

$$\frac{\delta d\theta}{d\theta} = \frac{M}{ES} \tag{13}$$

and after substituting it to the Eq. (2)

$$\sigma = \frac{y}{\rho}\frac{\delta d\theta}{d\theta} E = \frac{y}{\rho}\frac{M}{ES} E = \frac{y}{\rho}\frac{M}{S} \tag{14}$$

Therefore the design formula for the curved bars is

$$\sigma = \frac{My}{S\rho} \tag{15}$$

in which M is the bending moment in the analysed cross-section, S is a static moment that has to be calculated for a given shape of the cross-section, y is the distance from the neutral axis to the fibre at which the stress is to be calculated and ρ is the radius of this fibre. Usually the stresses are calculated at the outside and inside fibres which gives the maximum tensile and compression stresses. If the position of the neutral line, at which the stress equals zero, is known there is enough information to plot the stress distribution on the cross-section. The distribution has a hyperbolic shape.

References

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