Strength of Mechanical Constructions

Energy methods in mechanics

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Solving mechanical problems

There are two approaches to solve problems in mechanics

- vectorial approach
  - equation of equilibrium is written for vectorial quantities like stress or forces; from this equations the relation between the quantities is derived
- energy approach
  - internal energy of the system approach minimum value at equilibrium; by minimisation of the expression for energy one can find the equilibrium of the structure
- energy of mechanical systems is defined through the work done on these systems

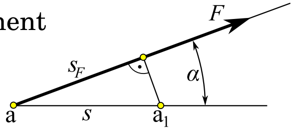
Equivalence of work and energy

Definition of work

- let's start with the definition of work
- the work done by force F on the point by moving it from position a to position a_1 is the product of the force and the displacement of the point
- since in general case the direction of force does not correspond with the direction of displacement we can write

$$W = F \cdot s \cos\alpha \quad \text{or} \quad W = F \cdot s_F$$

where s_F is the projection of displacement on the direction of the force



Equivalence of work and energy

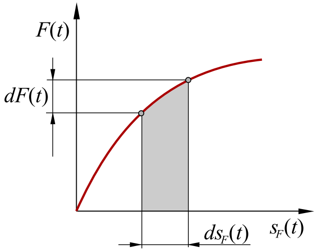
Definition of work

- in general case the force and the displacement depends on time
- having the relation of force and displacement in the form of a plot, one may determine the work by calculating the area under the plot
- elementary work will be equal to

$$dW = F(t) \cdot ds_F(t)$$

- and after integration

$$W = \int_0^{s_F} F(t) \cdot ds_F(t)$$



Equivalence of work and energy

How the potential energy of load is transformed into energy of elastic deformation?

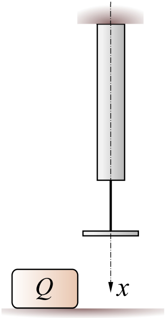
Assumptions:

- the load is a static one – is applied gradually and accelerations of structure can be omitted
- the deformation of the bar is slow enough to omit the inertia phenomenon
- at each point of loading the system is and its parts are in equilibrium
- material of the system follows the Hooke's law

Equivalence of work and energy

Transformation of the potential energy into energy of elastic deformation

- as an example let's consider the vertical bar fixed at the upper end with a rigid disc fixed at the other end
- next to the bar there is a weight Q, which will be placed on the disk
- after elevation the weight get some energy and after dropping on the disk it will lower because of the elongation of the bar

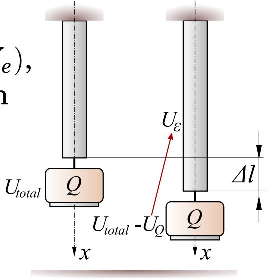


Equivalence of work and energy

Transformation of the potential energy into energy of elastic deformation

- lowering of the weight about Δl decreases its energy about U_Q
- this energy is transformed into energy of elastic deformation U_ε
- both energies can be expressed by the work – energy of the weight as the work of the external forces (W_e), and the energy of elastic deformation as a work of internal forces $(-W_i)$
- which is

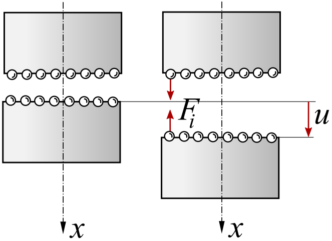
$$U_Q = W_e \qquad \text{and} \qquad U_\varepsilon = -W_i$$



Equivalence of work and energy

Transformation of the potential energy into energy of elastic deformation

- it should be explained why the work of internal forces is negative
- internal forces are the forces between atoms
- during elongation the atoms moves according to the axis of the bar but the forces are directed in the opposite way – thus the work is negative



Equivalence of work and energy

- according to the law of conservation of energy for elastic systems

$$U_\varepsilon = U_Q$$

- finally, because $U_Q = W_e$

$$U_\varepsilon = W_e$$

The above coincides with the principle of virtual work

$$W_e = -W_i \quad \rightarrow \quad W_e + W_i = 0$$

Attention!

only a different form of energy can change into potential energy of elastic deformation; the work W_e is only a measure of the potential energy of the external forces

First law of thermodynamics

- one can not measure the total amount of internal energy of a system
- however, one can measure the changes of this energy

First law of thermodynamics

The work performed on a mechanical system by external forces plus the heat that flows into the system from the outside equals the increase in internal energy plus the increase in kinetic energy

$$\delta W + \delta H = \delta U + \delta K$$

- assuming the lack of heat exchange and the movement

$$\delta W = \delta U$$

Energy of elastic deformation

Determination of elastic strain energy for simple case of load.

- tension (compression)
  - load: axial force F
    displacement: elongation Δl

$$U_F = \frac{1}{2}\frac{F^2 l}{EA}$$

$$U_F = \frac{1}{2}\int\limits_0^l \frac{F^2 dx}{EA} \quad or \quad U_F = \int\limits_0^l \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx$$

Energy of elastic deformation

Determination of elastic strain energy for simple case of load.

- torsion
  - load: torsional moment T
    displacement: angle of twist φ

$$U_T = \frac{1}{2}\frac{T^2 l}{GI_0}$$

$$U_T = \frac{1}{2}\int_0^l \frac{T^2 dx}{GI_0} \quad or \quad U_T = \int_0^l \frac{GI_0}{2}\left(\frac{d\varphi}{dx}\right)^2 dx$$

Energy of elastic deformation

Determination of elastic strain energy for simple case of load.

- bending
  - load: bending moment M
    displacement: angle of deflection θ

$$U_M = \frac{1}{2}\frac{M^2 l}{EI}$$

$$U_M = \frac{1}{2}\int_0^l \frac{M^2 dx}{EI} \quad or \quad U_M = \int_0^l \frac{EI}{2}\left(\frac{d^2 v}{dx^2}\right)^2 dx$$

Energy of elastic deformation

- all above formulae have the same structure, which helps to remember them and the use for solving problems
- the structure is

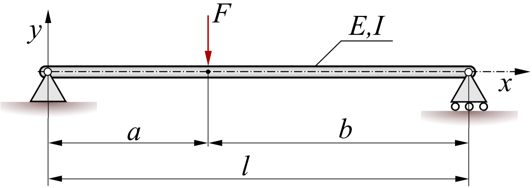
$$U = \frac{P_i \delta_i}{2}$$

  - δ_i – generalised displacement
    - $u, \ \varphi, \ \theta$
  - P_i – generalised force
    - $F, \ T, \ M$
- we will return to the above equation when the Castigliano's theorem will be derived

Energy of elastic deformation

Example 1

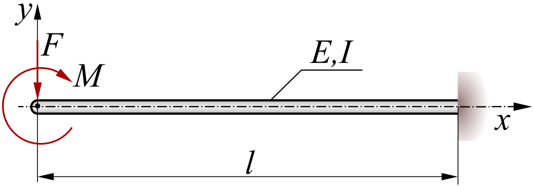
Determine the energy of elastic deformation stored in a simply supported beam loaded with transverse force F.



Energy of elastic deformation

Example 2

Determine the energy of elastic deformation stored in a cantilever beam loaded at one end with transverse force F and bending moment M.



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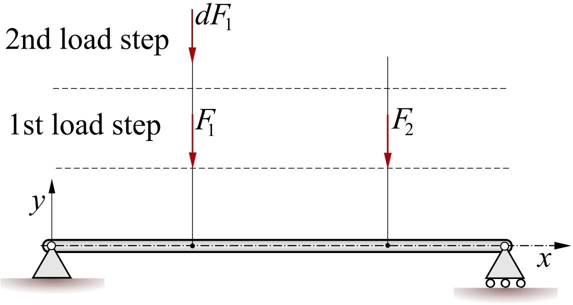
Castigliano's theorem

- consider a simply supported beam loaded with two steps
- energy of elastic deformation do not depends on the order of these load steps



Castigliano's theorem

Theorem

The partial derivative of the strain energy of a structure with respect to any load is equal to the displacement corresponding to that load.

$$\delta_i = \frac{\partial U}{\partial P_i}$$

- δ_i – generalised coordinate
  - $u,\ \varphi,\ \theta$
- P_i – generalised force
  - $F,\ T,\ M$

Castigliano's theorem

For convenience a modified Castigliano's theorem can be used

$$\delta_i = \frac{\partial U}{\partial P_i} = \frac{\partial}{\partial P_i} \int \frac{M(x)^2}{2EI} dx = \int \frac{M(x)}{EI} \frac{\partial M(x)}{\partial P_i} dx$$

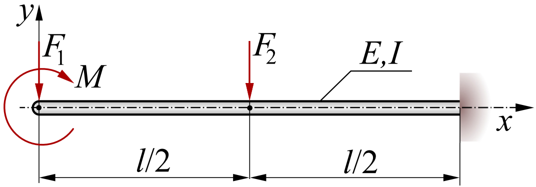
Since there are usually two generalised coordinates we can write

$$u = \frac{\partial U}{\partial F} = \int \frac{M(x)}{EI} \frac{\partial M(x)}{\partial F} dx$$

$$\theta = \frac{\partial U}{\partial M} = \int \frac{M(x)}{EI} \frac{\partial M(x)}{\partial M} dx$$

Castigliano’s theorem

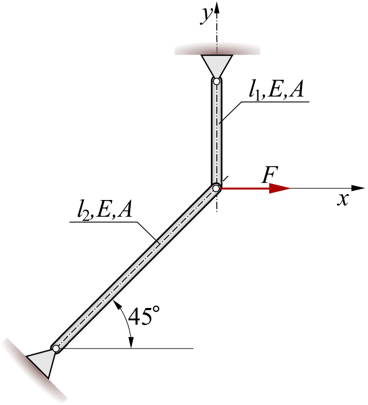
Determine the vertical displacement of the free end of the cantilever beam.



Castigliano's theorem

Example

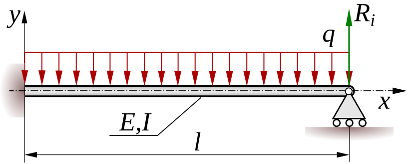
Determine horizontal u and vertical v displacement of the point where the load is applied.



Castigliano's theorem

- special case of Castigliano's theorem is Castigliano-Menabrea theorem
- it gives the possibility to determine statically indeterminate problems
- since it is known that the displacement of the point at the support is equal to zero one can write:

$$v_{R_i} = \frac{\partial U}{\partial R_i} = 0$$



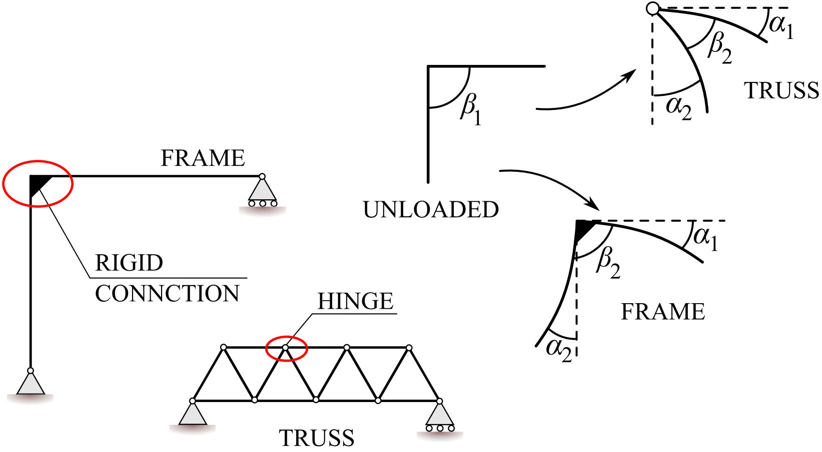
Castigliano’s theorem

Castigliano’s theorem gives the possibility to:

- determine the displacement of points of a structure at which load is applied; even complicated structures can be easily analysed
- determine the displacement of any point of the structure by applying a fictitious load
- solve statically indeterminate problems

Castigliano's theorem

Analysis of frames



Castigliano's theorem

The frame is loaded with load intensity q. Both elements of the frame are of equal length l and equal stiffness E, I.
Determine:

- maximum bending moment
- displacement of point C
- rotation at point B